Determining the Orientation of a Painted Sphere from a Single Image: A Graph Coloring Problem

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Abstract

In our work on robotic manipulation, we required a method for determining the orientation of a marked sphere from a single visual image. Our solution utilizes features of different colors painted on the sphere at the vertices of the Platonic solids. The main result of this paper is the minimization of the number of feature colors needed to solve the correspondence problem. The minimization of colors is a graph coloring problem.

Index Terms — Graph coloring, feature correspondence, Platonic solids, surface markings, sphere orientation.

1 Introduction

We are interested in the following problem: Given a sphere of known radius at a known location with an unknown orientation, paint the sphere with feature points so the orientation of the sphere can be determined from a single visual image.

This problem arose in our research on graspless robotic manipulation, such as throwing, catching, and controlling the orientation of a rolling ball. For feedback control of the orientation of a ball, we decided to paint the ball with feature points and use vision feedback from a single camera [3]. Assume the features are distinguished by their different colors. (Different shapes or sizes would also work.) Then the question is: Where should the feature points be placed on the sphere, and how many different feature colors are necessary?

To answer this question, we make the following three observations. (1) Assuming a calibrated camera and a known radius and location of the center of the sphere, the spatial location of a visible feature point can be determined by simply intersecting the line-of-sight of the image location with the known sphere surface. (2) The spatial locations of two feature points are sufficient to determine the orientation: one point (along with the known center point of the sphere) fixes an axis of rotation of the sphere, and the second determines the amount of rotation about that axis. If more than two points are visible, standard algorithms (e.g., [4]) can be used to find the best-fit orientation to the data.[1] (3) If each feature point has a unique color, the correspondence problem (matching features in the model to the observed features) is trivial. Therefore, an answer to the question is to paint the feature points so at least two points are visible for any orientation of the sphere, and to give each point a unique color.

Our interest is in minimizing the number of feature colors necessary to solve the correspondence problem. If fewer colors (or shapes or sizes) are needed, then the features can be more easily distinguishable from each other, increasing the robustness of the system to problems such as varying lighting conditions, which changes the perceived color of the features. In this work, we minimize the number of feature colors by exploiting the information in the known spatial relationship of the feature points. For instance, a sphere may have two blue feature points, one of which appears near red, yellow, and green features, and one of which appears near black, orange, and purple features. Then a blue feature in an image can be disambiguated by the colors of its neighboring features.

While many patterns of features over the sphere could be used, we chose to paint the sphere with features at the vertices of one of the five maximal inscribed Platonic solids (Figure 1). This assures equal distribution of feature points over the sphere. Treating the vertices and edges of the polyhedra as nodes and edges of graphs, we obtain the *correspondence graphs* in Figure 2. Defining n to be the number of vertices of the graphs, we have $n = 4$ for the tetrahedron (4 faces), $n = 6$ for the octahedron (8 faces), $n = 8$ for the cube (6 faces), $n = 12$ for the icosahedron (20 faces), and $n = 20$ for the dodecahedron (12 faces). The n vertices of the correspondence graph are colored with m colors.

Two problems need to be solved:

(i) *Finding the visible feature subgraph.* Many of the feature points will be occluded, so only a subgraph of the full correspondence graph will be visible. The visible subgraph depends on the orientation of the ball and the distance of the ball from the camera.

[1] In this paper, we are not concerned with the accuracy or error properties of any particular fitting algorithm.

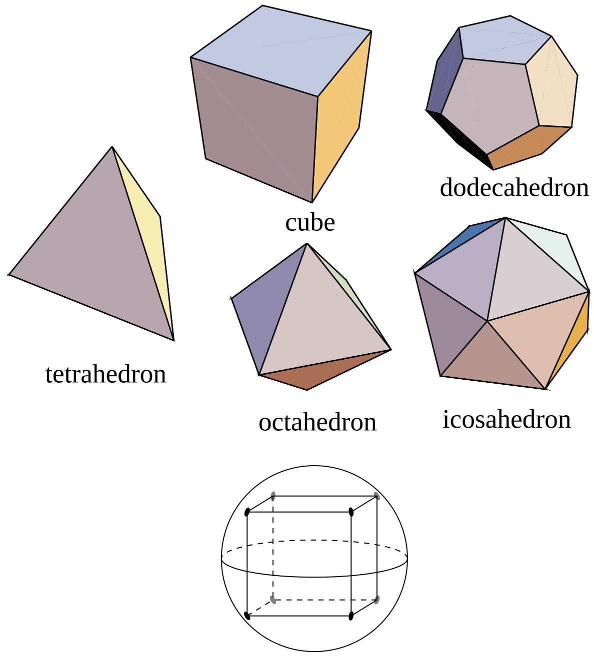


Figure 1: The five Platonic solids, and marking a sphere with feature points at the vertices of the maximal inscribed cube.

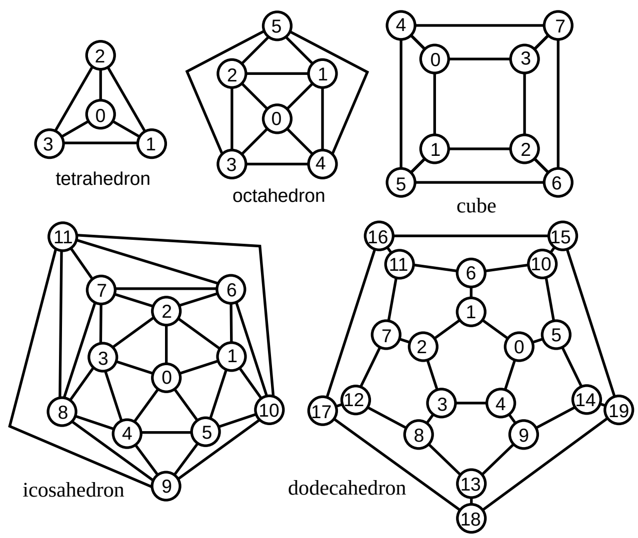


Figure 2: The correspondence graphs for the Platonic solids.

(ii) *Uniquely coloring the subgraphs.* Instead of uniquely coloring each feature point, it is only necessary to uniquely color each unique visible subgraph in the full graph to solve the correspondence problem. We would like to minimize the number of colors needed to solve this

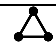
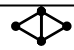
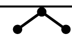
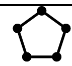
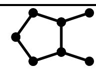
n	α	visible subgraph	rotational symmetries	unique colorings	instances in full graph	r lower bound	r
8	$> 54.74°$		1	m	8	–	–
8	$> 70.53°$		2	$(m^2 - m)/2$	12	6	7
12	$> 37.38°$		1	m	12	–	–
12	$> 63.43°$		3	$(m^3 - m)/3$	20	4	5
12	$> 79.19°$		2	$(m^4 + m^2 - 2m)/2$	30	3	4
20	$> 37.38°$		1	m	20	–	–
20	$> 41.81°$		2	$(m^2 - m)/2$	30	9	10
20	$> 54.74°$		1	m^3	60	4	5
20	$> 70.53°$		5	$(m^5 - m)/5$	12	3	3
20	$> 79.19°$		1	m^7	60	2	3

Table 1: For the viewing angles α indicated, the image is guaranteed to contain the subgraphs shown. "Rotational symmetries" indicates the number of indistinguishable orientations of the subgraph. The number of unique subgraph colorings are given, if m colors are used, as well as the number of instances of the subgraph in the full graph. Based on these, it is straightforward to derive a lower bound on r. The rightmost column represents the main contribution of the paper — the minimum number of different colors r needed to solve the correspondence problem. In most rows, r is greater than the lower bound due to coloring constraints imposed by the topology of the correspondence graph.

problem.

The second problem is a graph coloring problem. In this paper, we give the minimum number of different feature colors r needed to solve the correspondence problem as a function of the Platonic solid used to generate the features and the visual field of the camera (how much of a hemisphere is visible). A related problem in graph theory is finding the *local distinguishing number* of a cyclic graph: the minimum number of colors such that there are no two isomorphic subgraphs (of a specified size) in the full graph [1].

The main results of this paper are summarized in Table 1. The rest of the paper describes the contents of the table.

Our results were developed for tracking the orientation of a rotating ball being manipulated by a robot. A closely related problem is that of designing an absolute encoder for a spherical

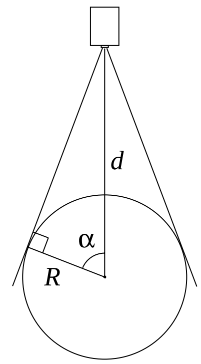


Figure 3: The visual field of the camera.

motor [2, 5]. Scheinerman *et al.* [6] have proposed a method for determining the orientation of a randomly painted black-and-white sphere using many photoreflective sensors scattered around the outside of the sphere, each returning a value of 0 or 1 indicating whether the sensor sees black or white surface. Based on precise knowledge of how the sphere was painted, they propose an iterative optimization method that quickly converges to an estimate of the orientation of the sphere. In their experiments, error was on the order of 2 degrees for 50 sensors and 0.5 degrees for 200 sensors.

2 Visible Subgraphs

A pinhole camera is placed a distance d from the center of the radius R sphere (Figure 3). The visible angle of the sphere $\alpha = \tan^{-1}\sqrt{d^2/R^2 - 1} < 90^\circ$ indicates the amount of the hemisphere that is visible. As α increases, more features become visible.

Table 1 indicates the visible subgraph topology (vertex features and implied edges) as α increases for the cube, icosahedron, and dodecahedron. The values of α given in the table are the minimum values at which the image is guaranteed to contain the given subgraph for any orientation of the sphere. (For any *particular* orientation of the sphere, more features may be visible. The subgraphs given in the table are those that are guaranteed to be common for all sphere orientations.) For both the tetrahedron ($n = 4$) and the octahedron ($n = 6$), there is no guarantee that more than one feature will be visible in the image for any $\alpha < 90^\circ$. Since at least two features are required to determine the orientation of the sphere, we will not consider the tetrahedron or octahedron further.

The visible subgraphs are not quite the same as the visible faces in the perspective aspect graphs of the Platonic solids, as self-occlusion occurs for the sphere where it does not occur for a polyhedron.

3 Colorings

If the k vertices of a subgraph are colored with m colors, the subgraph will have m^k unique colorings, minus any duplicates due to symmetry. For instance, if the colors 1 and 2 are used to color the three vertices of an equilateral triangle, then the colorings 1-2-2, 2-1-2, and 2-2-1 (rotated versions of each other) are indistinguishable. (The orientation of the sphere is not known in advance; this is what we are trying to solve.) Similarly, the coloring 1-1-1 is not a valid coloring, since the vertices are impossible to distinguish from each other. The number of unique subgraph colorings in Table 1 can be obtained by an application of Burnside's lemma.

The lower bound on r in Table 1 is obtained by finding the smallest number of colors m such that the number of unique colorings of a visible subgraph is greater than or equal to the number of such subgraphs in the full correspondence graph. If m is greater than or equal to this lower bound, then it may be possible to color the correspondence graph so that each subgraph is distinctly recognizable in the image.

Coloring the full graph places strong constraints on which unique subgraph colorings can be used, however. By coloring one subgraph, we have constrained the possible colorings of adjacent subgraphs. Therefore, it is often the case that the required number of colors r is larger than the lower bound.

In the seven relevant cases in Table 1, r is equal to the lower bound in one case, and r is one greater than the lower bound in the remaining six cases. For three of these six cases, the Appendix gives simple proofs that the lower bound is not sufficient. For the remaining three cases, we wrote computer programs which enumerated all possible colorings using m equal to the lower bound and verified that none uniquely colored all visible subgraphs. We are working on more concisely stated proofs that the lower bounds do not suffice for these three cases.

Note that three is the fewest number of colors that allow the correspondence problem, for the dodecahedron with at least five vertices of a face visible.

For the values of r listed in Table 1, we have found colorings of the correspondence graphs that solve the correspondence problem. Some of these colorings can be found simply by hand, and others were found by computer programs. All colorings were verified by computer programs which checked that each visible subgraph was uniquely colored. Example correspondence graph colorings are given below, where the order of the colors corresponds to the vertex assignments in the graphs of Figure 2.

Cube (8 vertices)

$\alpha > 70.53°$: 7 colors 02314506

Icosahedron (12 vertices)

$\alpha > 63.43°$: 5 colors 221043132140
$\alpha > 79.19°$: 4 colors 321213201000

Dodecahedron (20 vertices)

$\alpha > 41.81°$: 10 colors 40123696917534820875
$\alpha > 54.74°$: 5 colors 41121302441300432203
$\alpha > 70.53°$: 3 colors 12012201211001212210

4 Conclusion

The spatial relationship of image features can provide valuable information in solving the feature correspondence problem. In this paper we used this information to fully solve the correspondence graph coloring problem for the special case of a sphere painted with features at the vertices of the Platonic solids.

Acknowledgments

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Appendix

Lemma 4.1. *For the inscribed cube ($n = 8$) with two vertices visible, the correspondence problem cannot be solved with $m = 6$ colors.*

Proof. If $m = 6$, then at least two colors are used twice, or one color is used three times. Assume two colors are used twice. The two vertices colored 0 must not be adjacent to each other, or else they will not be distinguishable when they are the only two vertices visible. Therefore they must be placed at opposite corners of the cube. Similarly, the two vertices colored 1 must also be at opposite corners of the cube. In this case, they will both be adjacent to a vertex colored 0, so this is not a valid coloring.

If one color is used three times, two vertices with the same color will be adjacent, which is not a valid coloring. The correspondence problem cannot be solved with $m = 6$ colors. □

Lemma 4.2. *For the inscribed icosahedron ($n = 12$) with a triangle (three vertices) visible, the correspondence problem cannot be solved with $m = 4$ colors.*

Proof. For $m = 4$, there are 20 possible colorings of the visible subgraph triangle by Table 1. Since there are 20 instances of the triangle in the icosahedron, each coloring must appear once in the correspondence graph. This means the triangles 0-0-1, 0-0-2, and 0-0-3 must appear. Therefore, the graph must contain two adjacent vertices colored 0. This forms one edge of two triangles, and the third vertex of these two triangles must be colored 1 and 2 to get the triangles 0-0-1 and 0-0-2. To get the triangle 0-0-3, another 0 must be placed adjacent to one of the first two 0's, since only one of the first two 0's can appear in a new triangle with two 0's. However, a 0 adjacent to one of the first two 0's will also be adjacent to the 1 or 2, resulting in another triangle 0-0-1 or 0-0-2. The correspondence problem cannot be solved with $m = 4$ colors. □

Lemma 4.3. *For the inscribed icosahedron ($n = 12$) with a double triangle (four vertices) visible, the correspondence problem cannot be solved with $m = 3$ colors.*

Proof is by enumeration of all possible colorings by computer.

Lemma 4.4. *For the inscribed dodecahedron ($n = 20$) with two vertices visible, the correspondence problem cannot be solved with $m = 9$ colors.*

Proof. If $m = 9$, at least two colors must appear three or more times. Assume the color 0 appears three times. The three 0's must be placed so they are not adjacent to each other, or else they would be indistinguishable when only two vertices colored 0 are visible. Each of the three vertices colored 0 must have three unique colors adjacent, so that each of the pairs is distinguishable. This implies nine other unique colors, in addition to the color 0, so the correspondence problem cannot be solved with $m = 9$ colors. □

Lemma 4.5. *For the inscribed dodecahedron ($n = 20$) with three vertices visible, the correspondence problem cannot be solved with $m = 4$ colors.*

Proof is by enumeration of all possible colorings by computer.

Lemma 4.6. *For the inscribed dodecahedron ($n = 20$) with seven vertices visible (vertices of one pentagonal face plus two vertices of an adjacent face), the correspondence problem cannot be solved with $m = 2$ colors.*

Proof is by enumeration of all possible colorings by computer.