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# Controllability of Pushing

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## Abstract

*This paper addresses the question "Can the object be pushed from here to there?" We characterize the set of objects that are controllable (can be positioned arbitrarily), with and without obstacles, for the cases of point and line pushing contact. For the case of line contact, we find a set of pushing directions that keep the object fixed to the pusher, and we use these pushing directions to find sensorless plans to reposition the object among obstacles.*

## 1 Introduction

A robotic manipulator is often required to move an object from one place to another. An obvious solution is to equip the manipulator with a gripper and adopt the pick-and-place approach. By designing the grasp to resist all forces that could reasonably act on the object during the motion, grasp planning and path planning can be decoupled.

If the object is too large to be grasped or too heavy to be carried, however, this approach fails. It underutilizes the resources available to the robot, as it considers only the control forces that can be statically applied at the gripper. In general, the manipulator can apply control forces through any of the frictional kinematic constraints that comprise the manipulator. Other useful sources of control forces include gravity, the frictional kinematic constraints (floor, walls, obstacles) making up the robot's environment, and dynamic forces. If the robot can reason about these forces, it can use a richer set of manipulation primitives, including pushing, throwing, and striking. One emphasis of our research is to study how the set of achievable tasks grows as we endow the robot with a better understanding of mechanics.

In this paper we examine the sufficiency of pushing for positioning objects in the plane. Because the object is not firmly grasped, the forces that can be applied are limited, and therefore the possible motions of the object are limited. The "grasp" (pushing contact configuration) and manipulator path cannot be decoupled. By modeling the frictional forces, however, we can find pushing plans to move the object from one configuration in the plane to another by simultaneously designing the pushing contact and manipulator path such that the contact resists all expected forces during the motion.

This paper defines classes of objects that can be arbitrarily positioned by pushing. We give necessary and sufficient conditions for an object to be pushed to any position in the obstacle-free plane. We also show that almost any object can follow any free path arbitrarily closely by pushing it with point contact. We then describe a procedure for finding pushing motions that maintain a stable line contact (object remains fixed to the pusher) when the center of friction of the object is known. This is used to find sufficient conditions for an object to be pushed to any configuration in the obstacle-free plane, along with sufficient conditions for an object to follow any path arbitrarily closely, by stable pushing with line contact. Finally, we demonstrate a planner that finds paths using stable pushes.

### 1.1 Related work

Mason [18] derived a simple decision procedure for determining the rotation sense of an object with a known center of mass pushed at a point contact. This result was used in planners to eliminate uncertainty in an object's orientation by Brost [6], Goldberg [10], and others. Peshkin and Sanderson [22] and Mason and Brost [19] extended this work by finding bounds on the rotation rate. Part mating by multiple contact pushing has been investigated by Brost [7]. The motion of sliding objects has been studied by Alexander and Maddocks [4] using no information about the distribution of support forces, and by Goyal *et al.* [11] using perfect information.

A particularly relevant result is that of Akella and Mason [2], which states that any polygonal object can be arbitrarily repositioned in the obstacle-free plane by linear pushes with a perfectly rough fence. This result demonstrates the sufficiency of a particular manipulation strategy for a class of parts. Related in spirit are characterizations of graspable objects by point fingers (Mishra *et al.* [20]) and parallel-jaw grippers (Brost [6]); the demonstration of the controllability of a ball rolling on a plane or another ball (Li and Canny [15]); and the classification of orientable parts by parallel-jaw grasping sequences (Goldberg [10]).

Pushing is a type of graspless manipulation. Other types include tray-tilting to orient planar parts (Erdmann and Mason [9]), tumbling (Sawasaki *et al.* [24]) and pivoting (Aiyama *et al.* [1]) objects on a support surface, and whole arm manipulation (Salisbury *et al.* [23]).

The set of possible motion directions for a pushed object specifies a set of nonholonomic motion constraints. To demonstrate the controllability of an object by pushing, we appeal to nonlinear control theory. A good introduction to this area is given by Nijmeijer and van der Schaft [21].

### 1.2 Assumptions

Friction conforms to Coulomb's law. The frictional force at a sliding contact opposes the motion with magnitude $\mu_k f_n$, where $f_n$ is the magnitude of the normal contact force and $\mu_k$ is the kinetic coefficient of friction. At a sticking contact,

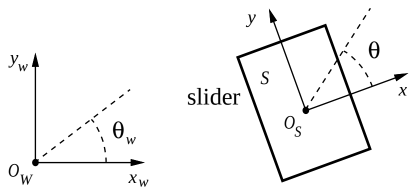



**Figure 1:** The world frame $\mathcal{F}_\mathcal{W}$ and the slider frame $\mathcal{F}_\mathcal{S}$.

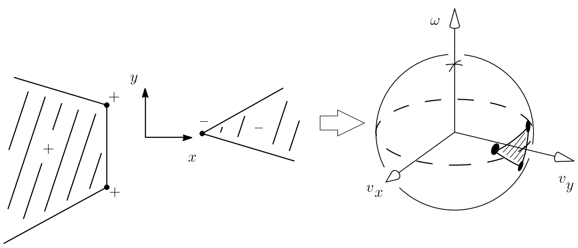



**Figure 2:** The convex hull of three velocity directions in the rotation center space and on the velocity sphere.

the frictional force can act in any tangential direction with any magnitude less than or equal to $\mu f_n$, where $\mu$ is the static coefficient of friction. For simplicity, we assume that the static and kinetic coefficients of friction are equal.

We assume that pushing motions are slow enough that inertial forces are negligible. This is the quasi-static assumption. Pushing forces lie in the horizontal support plane and are always balanced by the support frictional forces acting on the object. The support plane is a homogeneous surface, and gravity acts along the vertical.

### 1.3 Definitions

The slider $\mathcal{S}$ is a rigid object in the plane $\mathcal{W} = \mathbf{R}^2$, and its configuration space $\mathcal{C}$ is $\mathbf{R}^2 \times S^1$. The slider is pushed by a rigid pusher at a point or set of points on a closed, piecewise smooth curve $\Gamma$, which typically forms the perimeter of the slider $\mathcal{S}$. A world frame $\mathcal{F}_\mathcal{W}$ with origin $O_\mathcal{W}$ is fixed in the plane, and a slider frame $\mathcal{F}_\mathcal{S}$ with origin $O_\mathcal{S}$ is attached to the center of friction of the slider $\mathcal{S}$. (For a uniform coefficient of support friction, the center of friction of the slider is the point in the support plane beneath the center of mass.) The configuration $\mathbf{q} = (x_w, y_w, \theta_w)^T$ describes the position and orientation of the slider frame $\mathcal{F}_\mathcal{S}$ relative to the world frame $\mathcal{F}_\mathcal{W}$. See Figure 1.

Generalized forces $\mathbf{f}$ and velocities $\mathbf{v}$ are defined with respect to the slider frame $\mathcal{F}_\mathcal{S}$. A force $\mathbf{f} \in \mathbf{R}^3$ is given by its force and moment components $(f_x, f_y, m)^T$. A nonzero force $\mathbf{f}$ is the product of its scalar magnitude $f$ and its direction $\hat{\mathbf{f}} = (\hat{f}_x, \hat{f}_y, \hat{m})^T$. A force direction is a three-dimensional unit vector and may be represented as a point on the unit sphere ($\hat{\mathbf{f}} \in S^2$). The sphere of force directions is called the force sphere. Similarly, a nonzero velocity $\mathbf{v} = (v_x, v_y, \omega)^T$ is given by the product of its magnitude $v$ and direction $\hat{\mathbf{v}} = (\hat{v}_x, \hat{v}_y, \hat{\omega})^T$, and the sphere of velocity directions is called the velocity sphere. We will sometimes represent a velocity direction by its rotation center in $\mathcal{F}_\mathcal{S}$. The rotation center is the point about which the velocity is a pure rotation, along with the sense of rotation. Translations give rotation centers at infinity. Figure 2 illustrates the relationship between rotation centers and points on the velocity sphere.

For the quasi-static pushing problem, we are only concerned with force and velocity directions, not their magnitudes. We assume only that the manipulator is strong enough to move the slider, and that it moves slowly enough to satisfy the quasi-static assumption.

### 1.4 Overview

In the next section we define the pushing control system, some basic definitions of controllability, and their application to the pushing control system. Armed with these tools, in Section 3 we study the mechanics of pushing and the controllability of objects pushed with point contact or stable line contact. Section 4 demonstrates a planning algorithm for repositioning objects in the plane using stable pushes.

## 2 Controllability with velocity constraints

The set of velocity directions that the slider can follow during pushing is limited due to the limited set of force directions that can be applied by the pusher. In this section we study the controllability of planar objects subject to velocity constraints. We defer the problem of determining the motion of a pushed object to Section 3.

### 2.1 The pushing control system

The pushing control system can be described abstractly by the autonomous nonlinear control system $\dot{\mathbf{q}} = F(\mathbf{q}, \mathbf{c})$, where $\mathbf{c}$ is the control input describing the pushing contact configuration and the velocity of the pusher in the slider frame $\mathcal{F}_\mathcal{S}$. The motion of the slider in the world frame $\mathcal{F}_\mathcal{W}$ is a function $F$ of the control input and the configuration of the slider. For the rest of this paper, we will use the following more concrete description of the control system $\Sigma$:

$$\Sigma: \quad \dot{\mathbf{q}} = F(\mathbf{q}, \mathbf{c}_u) = X_u(\mathbf{q}),$$

$$\mathbf{q} \in \mathcal{C} = \mathbf{R}^2 \times S^1, \quad u \in \{0, \ldots, n\},$$

$$X_u(\mathbf{q}) = \begin{cases} (0,\ 0,\ 0)^T & \text{if } u = 0 \\ \begin{pmatrix} \cos\theta_w & -\sin\theta_w & 0 \\ \sin\theta_w & \cos\theta_w & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{v}_{ux} \\ \hat{v}_{uy} \\ \hat{\omega}_u \end{pmatrix} & \text{otherwise.} \end{cases}$$

A nonzero $u$ chooses one of $n$ distinct combinations of contact configurations and pushing velocities in the slider frame $\mathcal{F}_\mathcal{S}$. Associated with each control $\mathbf{c}_u$ is a vector field $X_u$ describing the motion of the slider in the world frame $\mathcal{F}_\mathcal{W}$. The tangent vector $X_u(\mathbf{q})$ is the (unit) velocity in the world frame $\mathcal{F}_\mathcal{W}$ of the slider at $\mathbf{q}$, and $\hat{\mathbf{v}}_u$ is the (unit) velocity of the slider in the slider frame $\mathcal{F}_\mathcal{S}$. The set of nonzero vector fields $X_u$ is denoted $\mathcal{X}$, and the set of nonzero velocity directions $\hat{\mathbf{v}}_u$ in the slider frame $\mathcal{F}_\mathcal{S}$ is $\mathcal{V}$. Each of the $n$ nonzero controls results in a distinct velocity direction.

Three aspects of the control system $\Sigma$ bear mentioning. (1) The absence of a drift vector field implies that the slider will not move when it is not pushed ($u = 0$). (2) For any constant control, the slider's velocity direction is constant in the slider frame $\mathcal{F}_{\mathcal{S}}$. (3) The control system is not necessarily symmetric. In general, it is not possible to follow a vector field $X_u$ backwards. This is due to the unilateral contact constraint.

## 2.2 Definitions of controllability

The pushing control system $\Sigma$, or equivalently the configuration of the slider $\mathcal{S}$, is *controllable from* $\mathbf{q}$ if, starting from $\mathbf{q}$, the slider can reach any point in the configuration space $\mathcal{C}$. The slider is *small-time locally controllable from* $\mathbf{q}$ if, for any neighborhood $U$ of $\mathbf{q}$, the set of reachable configurations without leaving $U$ contains a neighborhood of $\mathbf{q}$. The slider is *accessible from* $\mathbf{q}$ if the set of reachable configurations from $\mathbf{q}$ has nonempty interior in $\mathcal{C}$. The slider is *small-time accessible from* $\mathbf{q}$ if, for any neighborhood $U$ of $\mathbf{q}$, the set of reachable configurations without leaving $U$ has nonempty interior. (Although the phrase "small-time" appears in these terms, time does not appear in their definitions as they are applied in this paper.)

If a property holds for all $\mathbf{q} \in \mathcal{C}$, the phrase "from $\mathbf{q}$" can be omitted. For the control system $\Sigma$, any property that holds for any $\mathbf{q}$ also holds for all $\mathbf{q}$. Similarly, any property that does not hold for some $\mathbf{q}$ does not hold for any $\mathbf{q}$. As we will see, the conditions for controllability, accessibility, and small-time accessibility are identical for the system $\Sigma$, and small-time local controllability implies all of these. If the slider $\mathcal{S}$ is small-time locally controllable, it can follow any path in $\mathcal{C}$ arbitrarily closely.

Small-time accessibility can be established by an algebraic test on the set of vector fields $\mathcal{X}$. The Lie algebra $L(\mathcal{X})$ of the vector fields $\mathcal{X}$ is the space of linear combinations of these vector fields and the vector fields created by repeated Lie bracket operations. The Lie bracket of the vector fields $X$ and $Y$ is denoted $[X, Y]$. Defining $B_0(\mathcal{X}) = \mathcal{X}$ and $B_{k+1}(\mathcal{X}) = B_k(\mathcal{X}) \cup \{[X, Y]$ for all $X, Y \in B_k(\mathcal{X})\}$, the Lie algebra $L(\mathcal{X})$ is spanned by vector fields in $B_\infty(\mathcal{X})$. A control system is small-time accessible from $\mathbf{q}$ if it satisfies the *Lie Algebra Rank Condition*, which states that the tangent vectors at $\mathbf{q}$ of vector fields in $L(\mathcal{X})$ must span the tangent space at $\mathbf{q}$ (see, for example, Hermann and Krener [12]).

The Lie bracket $[X, Y]$ of the vector fields $X$ and $Y$ in local coordinates is

$$[X, Y](\mathbf{q}) = \frac{\partial Y(\mathbf{q})}{\partial \mathbf{q}} X(\mathbf{q}) - \frac{\partial X(\mathbf{q})}{\partial \mathbf{q}} Y(\mathbf{q}).$$

(See Nijmeijer and van der Schaft [21] for a derivation.) Using the definition of $X_u$ from Section 2.1, $\partial X_u(\mathbf{q})/\partial \mathbf{q}$ evaluates simply to

$$\begin{pmatrix} 0 & 0 & -\hat{v}_{ux} \sin\theta_w - \hat{v}_{uy} \cos\theta_w \\ 0 & 0 & \hat{v}_{ux} \cos\theta_w - \hat{v}_{uy} \sin\theta_w \\ 0 & 0 & 0 \end{pmatrix}.$$

For the control system $\Sigma$, the Lie algebra $L(\mathcal{X})$ is spanned by vector fields in $B_1(\mathcal{X})$: the distribution defined by $B_1(\mathcal{X})$ is involutive. We need only look at the vector fields $\mathcal{X}$ and their Lie brackets in order to decide small-time accessibility.

If the control system is symmetric (all vector fields can be followed forward and backward), then small-time accessibility implies small-time local controllability.

## 2.3 Controllability of the pushing control system

If $n = 1$ for the control system $\Sigma$, then the slider is confined to a one-dimensional integral curve of $X_1$, and the control system $\Sigma$ is not accessible. If $n = 2$, the Lie algebra $L(\mathcal{X})$ is spanned by $X_1$, $X_2$, and $X_3 = [X_1, X_2]$. If the determinant of the matrix $(X_1 \;\; X_2 \;\; X_3)$ is nonzero, then its rank is three. A simple calculation yields

$$\det(X_1 \;\; X_2 \;\; X_3) = (\hat{\omega}_2 \hat{v}_{1x} - \hat{\omega}_1 \hat{v}_{2x})^2 + (\hat{\omega}_2 \hat{v}_{1y} - \hat{\omega}_1 \hat{v}_{2y})^2.$$

The determinant is only zero if (1) $\hat{\omega}_1 = \hat{\omega}_2 = 0$ or (2) $\hat{\mathbf{v}}_1 = -\hat{\mathbf{v}}_2$. If condition (1) holds, the slider cannot rotate. If condition (2) holds, the slider is confined to a one-dimensional curve of the configuration space $\mathcal{C}$. If neither of these conditions hold, the Lie bracket operation has essentially created a new linearly independent control vector field and the Lie Algebra Rank Condition is satisfied.

**Proposition 1** *The control system $\Sigma$ is small-time accessible if and only if the set of velocity directions $\mathcal{V}$ contains two velocity directions, $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$, such that they are not both translations ($\hat{\omega}_1 \neq 0$ or $\hat{\omega}_2 \neq 0$) and $\hat{\mathbf{v}}_1 \neq -\hat{\mathbf{v}}_2$. These conditions are also necessary and sufficient for accessibility and controllability.*

**Proof:** The proof that these conditions are necessary and sufficient for small-time accessibility is given above, and accessibility follows directly. Regarding controllability, consider the following two cases. If $\hat{\mathbf{v}}_2$ is a translation, the slider can reach any configuration by first rotating following $X_1$, then translating along $X_2$, and then rotating along $X_1$. If neither velocity direction is a translation, then, by alternating equal rotations along $X_1$ and $X_2$, the slider can move to any point in the plane with no net rotation. Then the slider can be rotated to the desired goal configuration. □

Proposition 1 is a straightforward generalization of a result due to Barraquand and Latombe [5] which states that the Lie Algebra Rank Condition is satisfied for any car-like mobile robot that can take at least two steering angles. A car-like mobile robot can drive both forward and backward, and this symmetry, coupled with small-time accessibility, implies small-time local controllability. As we have already noted, however, the pushing control system $\Sigma$ may not be symmetric, so small-time local controllability does not follow from small-time accessibility. Before addressing the conditions for small-time local controllability, we establish the following fact.

**Proposition 2** *Consider a set of velocity directions $\mathcal{V}$ and its convex hull $\mathcal{V}_{CH}$ on the velocity sphere. Any path from $\mathbf{q}_1$ to $\mathbf{q}_2$ using velocity directions in $\mathcal{V}_{CH}$ can be followed arbitrarily closely by another path, also from $\mathbf{q}_1$ to $\mathbf{q}_2$, using only velocity directions in $\mathcal{V}$.*

**Proof:** Proposition 5 in Appendix B of (Barraquand and Latombe [5]) proves the case when $\mathcal{V}$ consists of two velocity directions. The result for any number of velocity directions follows by induction. □

On any open set of the configuration space $\mathcal{C}$, Proposition 2 says that we can consider the available velocity directions to be the convex hull of the velocity direction set $\mathcal{V}$. Therefore, if $\mathcal{V}$ contains four velocity directions that positively span the velocity sphere, the configuration of the slider $\mathcal{S}$ is small-time locally controllable. This also follows from Theorem 1:

**Theorem 1** (Sussmann [25]) *Let $\mathcal{X}$ be a finite set of vector fields on an open set of the state manifold containing $\mathbf{q}$. The set of nonzero tangent vectors at $\mathbf{q}$ is denoted $\mathcal{X}(\mathbf{q})$. Then*
*(1) If $\mathbf{0}$ is in the interior of the convex hull of $\mathcal{X}(\mathbf{q})$, the system is small-time locally controllable from $\mathbf{q}$.*
*(2) If $\mathbf{0}$ does not belong to the convex hull of $\mathcal{X}(\mathbf{q})$, the system is not small-time locally controllable from $\mathbf{q}$.*

The first half of Theorem 1 indicates that the control system $\Sigma$ is small-time locally controllable if the velocity direction set $\mathcal{V}$ positively spans the velocity sphere. The second half of the theorem says that $\Sigma$ is not small-time locally controllable if $\mathcal{V}$ is confined to any open hemisphere of the velocity sphere.

The only remaining case is when $\mathcal{V}$ is confined to a closed hemisphere, but not an open hemisphere, of the velocity sphere. To resolve this case, we must consider the derivatives of $\mathcal{X}(\mathbf{q})$ (Sussmann [25]). We conclude that $\Sigma$ is small-time locally controllable if and only if $\mathcal{V}$ positively spans a great circle of the velocity sphere that does not lie in the $\omega = 0$ plane. To see this is sufficient, recall that two nonopposite velocity directions that are not both translations are sufficient for small-time accessibility. If both velocity directions can be reversed (a total of four velocity directions), then the system is small-time locally controllable. These velocity directions positively span a great circle of the velocity sphere such that $\hat{\omega}$ is not identically zero. By Proposition 2, on any open set of the configuration space $\mathcal{C}$, any set of velocity directions that span the same great circle is equivalent.

**Proposition 3** *The control system $\Sigma$ is small-time locally controllable if and only if the set of velocity directions $\mathcal{V}$ positively spans a great circle of the velocity sphere that does not lie in the $\omega = 0$ plane.*

Figure 3 gives examples of rotation center sets that yield small-time local controllability.

Corollary 1 follows directly from Propositions 1 and 3.

**Corollary 1** *The number of distinct combinations $n$ of pushing contact configurations and pushing directions must be (1) at least two for the configuration of the slider $\mathcal{S}$ to be controllable by pushing, and (2) at least three for the configuration of the slider $\mathcal{S}$ to be small-time locally controllable by pushing. These bounds are tight.*

# 3 Mechanics and controllability of pushing

In this section we study the problem of determining the motion of a pushed object. Using the results of the previous section, we elucidate the controllability properties of objects pushed with either point contact or stable line contact.

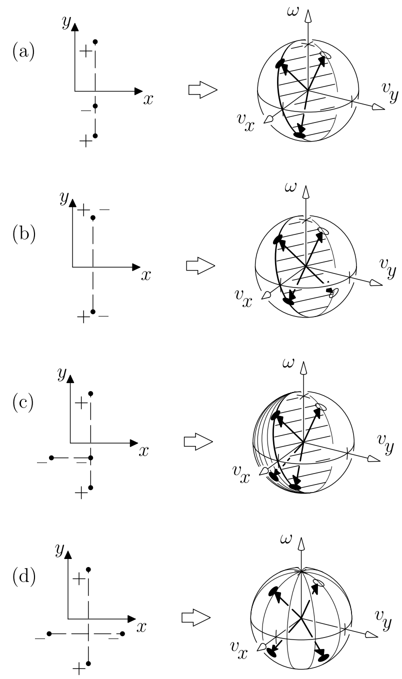



**Figure 3:** Examples of rotation center sets that yield small-time local controllability: (a) three rotation centers positively spanning a great circle of the velocity sphere; (b) four rotation centers positively spanning a great circle; (c) four rotation centers positively spanning a hemisphere; (d) four rotation centers positively spanning the entire velocity sphere.

## 3.1 Mechanics of pushing

### 3.1.1 The limit surface

During quasi-static pushing, the force $\mathbf{f}$ applied by the pusher is equal to the force applied by the slider to the support plane. This force is a function of the slider's velocity direction $\hat{\mathbf{v}}$ and its support friction distribution $s(\mathbf{x})$, where $s(\mathbf{x})$ is the product of the nonnegative support pressure and friction coefficient at each point $\mathbf{x}$ of the support area.

This function can be given a convenient geometric interpretation. As the slider's velocity direction $\hat{\mathbf{v}}$ moves over the velocity sphere, the force $\mathbf{f}$ moves on a two-dimensional surface in the three-dimensional force space. This closed, convex surface is called the *limit surface* (Goyal *et al.* [11]). The limit surface encloses the set of all forces that can be statically applied to the slider, and during quasi-static motion the applied force lies on the limit surface. The slider's velocity direction vector $\hat{\mathbf{v}}$ is normal to the limit surface at the force $\mathbf{f}$ (Figure 4). If the force $\mathbf{f}$ lies on the limit surface with an associated velocity direction $\hat{\mathbf{v}}$, then the force $-\mathbf{f}$ also lies on the limit surface with an associated velocity direction $-\hat{\mathbf{v}}$.

If the support friction distribution $s(\mathbf{x})$ is finite everywhere, the limit surface is smooth and strictly convex, defining a continuous one-to-one mapping from the set of force directions $\hat{\mathbf{f}} \in S^2$ to the set of velocity directions $\hat{\mathbf{v}} \in S^2$. If the applied force has zero moment about the center of friction (the centroid of $s(\mathbf{x})$), the resulting slider velocity is translational and parallel to the applied force.

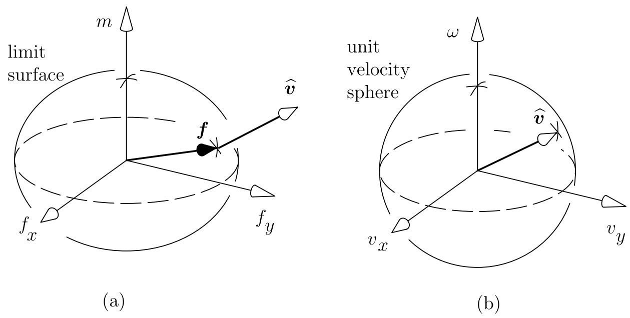



**Figure 4:** Limit surface mapping of a force to a velocity direction.

If the support friction distribution $s(\mathbf{x})$ becomes infinite at any point, however, the mapping is no longer one-to-one. For example, if a point $\mathbf{x}_0$ supports a finite force, the pressure at $\mathbf{x}_0$ is infinite, and if the coefficient of friction is nonzero, then $s(\mathbf{x}_0)$ is infinite. In these cases, the limit surface contains flat facets. At these facets, a set of force directions maps to the same velocity direction (rotation about the support point).

If the support friction distribution $s(\mathbf{x})$ is infinite only at points on a line, and $s(\mathbf{x})$ integrates to zero over the rest of the support surface, then the limit surface contains vertices. The normals to the limit surface at these vertices are not uniquely defined: the same force maps to a set of possible velocity directions.

### 3.1.2 Solving for the motion of a pushed object

Each contact point between the pusher and the slider may be sticking, breaking free, or sliding to the left or right. The *contact mode* describes the qualitative behavior of each contact point between the pusher and the slider. For each possible contact mode $i$, there is a space $V_{k,i}$ of slider velocities that are kinematically consistent with that contact mode and the known pusher velocity (Lynch [16]). By Coulomb's law, each contact mode also specifies a polyhedral cone of possible pushing forces in the three-dimensional force space. This cone is the convex hull of the individual friction cones at the sticking contacts and the friction cone edges at the sliding contacts (Erdmann [8]). This composite friction cone is intersected with the limit surface to find a cone of possible velocities $V_{f,i}$ (Figure 5). If $V_{k,i} \cap V_{f,i} = \emptyset$, contact mode $i$ cannot occur; otherwise, contact mode $i$ is feasible and any of the velocities in the intersection set is a possible solution to the motion of the slider.

## 3.2 Controllability with point contact pushing

Several researchers have constructed feedback control systems for point contact pushing. Here we examine the controllability of such a system.

**Proposition 4** *The configuration of a slider $\mathcal{S}$ with a bounded support friction distribution $s(\mathbf{x})$ is controllable by pushing if and only if the pusher can apply two pushing force directions, $\hat{\mathbf{f}}_1$ and $\hat{\mathbf{f}}_2$, such that they do not both pass through the center of friction ($\hat{m}_1 \neq 0$ or $\hat{m}_2 \neq 0$) and $\hat{\mathbf{f}}_1 \neq -\hat{\mathbf{f}}_2$.*

**Proof:** Because $s(\mathbf{x})$ is bounded, the limit surface has no facets, and therefore the two force directions map through

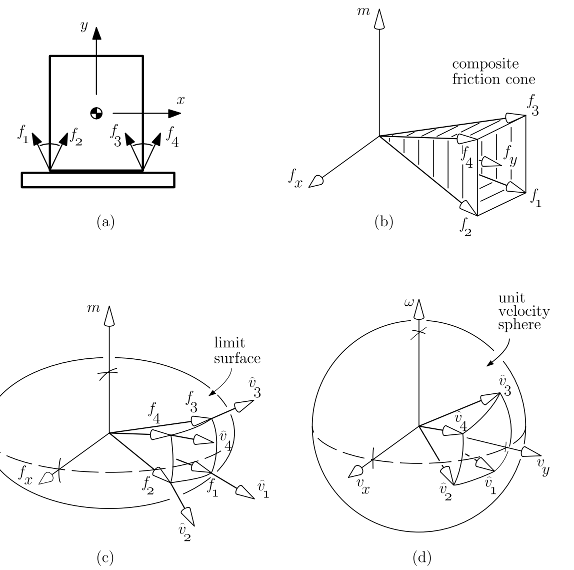



**Figure 5:** (a) The forces that the pusher can apply to the slider during sticking contact are represented by the two friction cones. (b) The convex hull of these friction cones in the three-dimensional force space. The result is a composite friction cone of possible pushing forces. (c) Mapping these forces through the limit surface for the slider. (d) The slider velocity directions corresponding to forces on or inside the composite friction cone.

the limit surface to two distinct velocity directions. At least one of the force directions has nonzero moment, so at least one of the velocity directions has a nonzero angular component. Because the two force directions are not opposite, the two velocity directions are also not opposite. Therefore, by Proposition 1, the slider is controllable. □

An uncontrollable slider is a disk centered at its center of friction with a pushing friction coefficient of zero. All pushing forces pass through the center of friction, creating zero moment about the center of friction. The slider cannot be rotated (unless its limit surface contains vertices).

If the slider is polygonal, a natural question is whether the slider is controllable by pushing with point contact on a single edge. Theorem 2 is a direct application of Proposition 4.

**Theorem 2** *The configuration of a slider $\mathcal{S}$ with a bounded support friction distribution $s(\mathbf{x})$ is controllable by pushing on a straight edge if and only if the edge (1) has nonzero length or (2) has nonzero friction and is not a point at the center of friction.*

A slider that is controllable by pushing may have to be pushed a long distance to reach nearby configurations. If the object is small-time locally controllable, however, it can follow any path arbitrarily closely. In order to find conditions for small-time local controllability of a slider, first recall that the set of available pushing contacts is given by $\Gamma$, a closed, piecewise smooth curve. At each point of $\Gamma$ that is not a vertex, the curve $\Gamma$ has a unique inwardly-pointing contact normal. At a vertex, we assume that the contact normal can take any direction in the range specified by the contact normals adjacent to the vertex. Each contact point and contact normal specifies a pushing force direction that can be applied to the frictionless slider $\mathcal{S}$, and the curve of all such force directions is denoted $\hat{\mathbf{f}}(\Gamma)$. Because $\Gamma$ is a closed curve, $\hat{\mathbf{f}}(\Gamma)$ is a (possibly self-intersecting) closed curve of force directions on the force sphere.

**Theorem 3** *The configuration of any slider $\mathcal{S}$ with a closed, piecewise smooth curve $\Gamma$ of available pushing contact points is small-time locally controllable by pushing with point contact, unless the pushing contact is frictionless and $\Gamma$ is a circle centered at the center of friction (a frictionless disk).*

**Proof:** *Case 1: $\Gamma$ not a circle.* Following the argument of Hong *et al.* [13], $\hat{\mathbf{f}}(\Gamma)$ must contain at least two pairs of opposite force directions. By the limit surface mapping, these forces yield two pairs of opposite velocity directions that span a great circle of the velocity sphere. By Proposition 3, the slider is small-time locally controllable, unless this great circle lies in the $\omega = 0$ plane. In this case we use the result of Mishra *et al.* [20] which states that $\hat{\mathbf{f}}(\Gamma)$ positively spans the force sphere. Therefore $\hat{\mathbf{f}}(\Gamma)$ contains forces with positive and negative moment, and the slider can be rotated clockwise or counterclockwise. The $\omega = 0$ great circle and any clockwise and counterclockwise directions positively span the velocity sphere, and the slider is small-time locally controllable.

*Case 2: $\Gamma$ a circle.* Every pair of diametrically opposed points on $\Gamma$ gives rise to a pair of opposite velocity directions. If the center of friction is offset from the center of the circle, then only one pair lies in the $\omega = 0$ plane, and therefore any two pairs of opposite velocity directions yield small-time local controllability. If the center of friction is at the center of the circle and there is nonzero friction at the pushing contact, the slider can be translated in any direction and rotated using frictional forces to create moment about the center of friction. The slider is small-time locally controllable. If the contact is frictionless, however, the object cannot be rotated. A frictionless disk centered at its center of friction is the only type of slider that is not small-time locally controllable by point contact pushing. (If the slider's limit surface contains vertices, it may rotate nondeterministically.) □

## 3.3 Stable pushing with line contact

### 3.3.1 Mechanics of line contact pushing

In the previous section we examined the controllability of sliders pushed with point contact, but we would also like to synthesize pushing controllers. Unfortunately, the motion of a slider pushed with a single point of contact is often unpredictable, because it depends on the unknown and generally indeterminate support friction distribution $s(\mathbf{x})$. If there are two or more simultaneous pushing contacts, however, there may exist a space of pushing directions that, despite uncertainty in $s(\mathbf{x})$, result in a predictable motion of the slider: sticking at all contact points. The slider is effectively rigidly attached to the pusher. We call such a push a stable push, and we will use these stable pushes to execute open-loop pushing plans.

In this section we focus on stable pushing with line contact: all contacts between the pusher and the slider are collinear

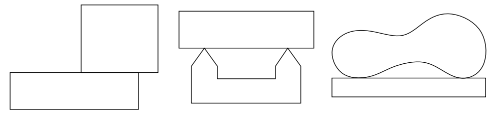


**Figure 6:** Examples of line contact.

with contact normals perpendicular to the line (Figure 6). We also assume that the coefficient of friction at all pushing contacts is the same. Further, the center of friction is known but the support friction distribution $s(\mathbf{x})$ is not.

For a given line contact, we use the following definitions:

$\mathcal{V}_{stable}$: The set of pushing directions such that the slider remains fixed to the pusher during the motion.

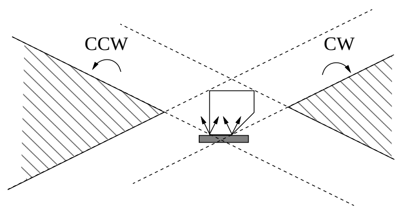
$\mathcal{V}_{\mathcal{F}}$: The set of pushing directions such that one solution to the motion of the slider is to remain fixed to the pusher. This set of velocity directions is found by intersecting the line contact composite friction cone $\mathcal{F}$ with the limit surface (Figure 5).

Although stable contact is always a possible solution if the pushing direction $\hat{\mathbf{v}}$ is in $\mathcal{V}_{\mathcal{F}}$, there may be other solutions. It is necessary to prove all other contact modes inconsistent. In this section we describe the procedure STABLE for finding a subset of $\mathcal{V}_{\mathcal{F}}$ and provide a theorem stating that this subset also belongs to $\mathcal{V}_{stable}$.

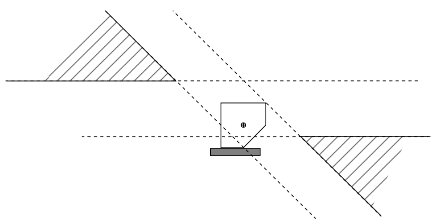
Figure 7 illustrates the procedure STABLE, which finds a subset of $\mathcal{V}_{\mathcal{F}}$ by combining results due to Mason and Brost [19], Peshkin and Sanderson [22], and Alexander and Maddocks [4]. (This is a much simpler, somewhat more conservative method than the computationally intensive algorithm we described in [16].) STABLE misses some tight-turning rotation centers but finds all translations belonging to $\mathcal{V}_{\mathcal{F}}$. If the composite friction cone $\mathcal{F}$ contains any pure force (zero moment about the center of friction) in its interior, STABLE will find a convex set of velocity directions with nonempty interior and a range of translation directions. If the composite friction cone $\mathcal{F}$ contains only a single pure force direction, necessarily on the boundary of $\mathcal{F}$, STABLE finds only a single translation in the direction of this pure force. If the composite friction cone $\mathcal{F}$ does not contain a pure force, STABLE finds no velocity directions belonging to $\mathcal{V}_{\mathcal{F}}$. In fact, with no information about the support friction distribution $s(\mathbf{x})$ other than the center of friction, no velocity direction is guaranteed to be in $\mathcal{V}_{\mathcal{F}}$ if the line contact cannot apply a force through the center of friction.

Proving that a pushing direction in the set found by STABLE belongs to $\mathcal{V}_{stable}$ requires proving all other contact modes inconsistent. Here we state the relevant result, omitting the proof by case analysis.

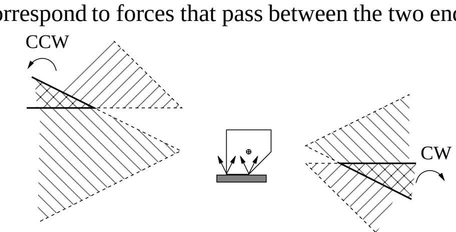
**Theorem 4** *Given a pusher in line contact with a slider $\mathcal{S}$ with a bounded support friction distribution $s(\mathbf{x})$, let $\mathcal{V}_{\mathcal{F}}$ be the set of rotation centers resulting from forces in the composite friction cone $\mathcal{F}$. Draw two lines perpendicular to the line contact such that the entire slider $\mathcal{S}$ is contained between the two lines. All rotation centers in $\mathcal{V}_{\mathcal{F}}$ and outside the two lines are guaranteed to belong to the set of stable pushing directions $\mathcal{V}_{stable}$. Therefore, all velocity directions found by* STABLE *belong to $\mathcal{V}_{stable}$.*




(a) The coefficient of friction (0.5 here) defines two friction cone edges, at angles $\tan^{-1}\mu$ to the contact normal, where $\mu$ is the coefficient of friction. For each edge of the friction cone, draw two lines perpendicular to the friction cone edge such that the entire slider is contained between the two lines. For an applied force at an edge of the friction cone, the resulting rotation center must lie in its respective band. Counterclockwise (clockwise) rotation centers between the two bands and to the left (right) of the slider correspond to force angles inside the angular limits of the friction cone.



(b) For each endpoint of the line contact, draw two lines perpendicular to the line through the center of friction and the endpoint. One of these lines is the perpendicular bisector between the contact point and the center of friction. The other is a distance $r^2/p$ from the center of friction and on the opposite side from the endpoint, where $p$ is the distance from the endpoint to the center of friction and $r$ is the distance from the center of friction to the most distant support point of the slider. (This "tip line" should actually be slightly more distant from the center of friction; see (Peshkin [22]) for details.) The rotation center from pushing on this endpoint must lie in the band between these two lines. All rotation centers between the two bands correspond to forces that pass between the two endpoints.




(c) The intersection of the closed regions found in (a) and (b) yield a set of rotation centers corresponding to forces that are guaranteed to lie on or inside the composite friction cone $\mathcal{F}$ from the line pushing contact.

**Figure 7:** Procedure STABLE.

### 3.3.2 Controllability by stable pushing with line contact

If the composite friction cone $\mathcal{F}$ from the line pushing contact contains a pure force in its interior, then STABLE finds a convex set of velocity directions with nonempty interior. By Proposition 1, we get the following.

**Theorem 5** *If a slider $\mathcal{S}$ is pushed with line contact with a composite friction cone $\mathcal{F}$ such that $\mathcal{F}$ contains a pure force (zero moment about the center of friction of the slider) in its interior, then the configuration of the slider is controllable by the stable pushes found by* STABLE.

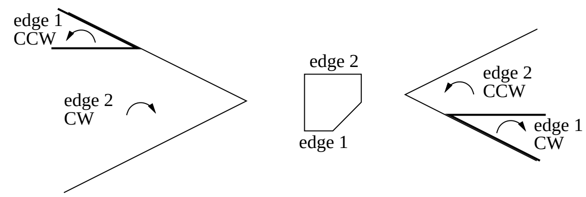



**Figure 8:** The slider of Figure 7 is small-time locally controllable by stable pushing at two edges. The friction coefficient is 0.5.

If Theorem 5 is satisfied, the slider can be pushed to any configuration in the obstacle-free plane using the stable pushes of STABLE. We can apply Theorem 5 to prove the following result regarding polygonal sliders.

**Theorem 6** *For a polygonal slider $\mathcal{S}$ and a sufficiently long straight-edge pusher, any edge of the convex hull of $\mathcal{S}$ can be used as the line pushing contact. If the center of friction of $\mathcal{S}$ does not lie on a vertex of the convex hull, and there is nonzero friction at the line contacts, then there is at least one line contact from which the slider is controllable by the stable pushes found by* STABLE.

**Proof:** The center of friction only lies on a vertex of the convex hull if the support friction distribution $s(\mathbf{x})$ integrates to zero everywhere else. Otherwise, there is at least one normal to the interior of an edge of the convex hull that passes through the center of friction. A force $\mathbf{f}$ along that normal is in the interior of the composite friction cone for that edge. By Theorem 5, the slider is controllable from that edge by the stable pushes of STABLE. □

If the pusher can change contact configurations, in some cases the configuration of the slider may be small-time locally controllable by pushing directions found by STABLE.

**Theorem 7** *Given a set of line pushing contacts with composite friction cones $\mathcal{F}_i$, find the set of all pure forces (zero moment about the center of friction of the slider) interior to at least one of the composite friction cones $\mathcal{F}_i$. If these pure forces positively span the plane of pure forces, then the configuration of the slider $\mathcal{S}$ is small-time locally controllable by the stable pushes found by* STABLE.

**Proof:** If the conditions of Theorem 7 are satisfied, then STABLE, applied to each line contact, finds a set of translations interior to $\mathcal{V}_{stable}$ that positively span the $\omega = 0$ great circle. Because each of these translations has a neighborhood of velocity directions also in $\mathcal{V}_{stable}$, the velocity directions found by STABLE positively span the velocity sphere. □

The slider of Figure 7 is small-time locally controllable by stable pushing with the two line contacts shown in Figure 8.

## 4 Planning pushing paths among obstacles

Stable pushes can be used in sensorless manipulation plans. Our planner is adapted from Barraquand and Latombe's path planner for nonholonomic mobile robots [5]. The approach is very simple. For each line contact, calculate

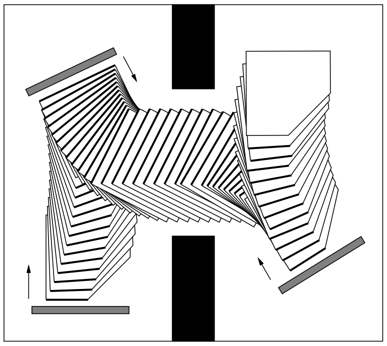


**Figure 9:** A plan found using stable pushes from only two edges.

the set of stable pushing directions using STABLE. In light of Proposition 2, the planner uses only the extremal velocity directions of each set. Given the discrete set of all extremal velocity directions, integrate each forward a small distance, and repeat the process from each new configuration while throwing away paths that return sufficiently near to a previous spot or result in collision of the pusher or slider with an obstacle. The paths are sorted by a user-specified cost function of the number of pushes, the number of changes of the pushing direction, and the number of changes in the pushing contact. The best-first search continues until the planner finds a minimum-cost path that reaches a specified neighborhood of the goal.

If the planar extent of the pusher is negligible and the slider is small-time locally controllable by stable pushes, then if a free path exists for the slider without any motion constraints, a pushing plan also exists. If a pushing plan exists using the stable pushes found by STABLE, then our planner, equipped with an exact collision-detection routine and suitable search parameters (e.g., the length of each pushing step), will find a plan. Unfortunately we do not know how to set the search parameters *a priori* to ensure this property. The planner and its properties are described in more detail in [17].

Figure 9 shows a pushing path for the slider of Figure 8. The planner, implemented in C on a DEC 5000, took four seconds to find this plan minimizing the contact changes. We assume that the pusher can be moved freely between pushes; if the pusher is confined to the plane of the obstacles, we must also plan the manipulator motions between contact configurations (Alami *et al.* [3], Koga [14]).

## 5 Conclusion

A model of the mechanics of a task is a resource for the robot, just as actuators and sensors are resources. The clever use of frictional, gravitational, and dynamic forces can substitute for extra actuators; the expectation derived from a good model can minimize sensing requirements. As the model becomes more detailed, however, it becomes more challenging to assess the capabilities of a robot. This paper addresses the capability of a pushing robot under a quasi-static model by elucidating some controllability properties.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