Distributed Estimation and Control of Swarm Formation Statistics

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***Abstract*— We describe distributed estimation algorithms that allow robots in a communication network to maintain estimates of summary statistics describing the shape of the swarm. We show that these estimators, combined with motion controllers implemented on each robot, result in the swarm formation statistics being driven to desired values in the presence of a changing network topology and the addition and deletion of robots.**

I. INTRODUCTION

We are interested in the following general problem: given a set of agents (e.g., mobile robots), design a control law to run on each agent, based on sensor and communication input, to achieve a desired collective "emergent" global behavior of the system. In other words, the global dynamical system defined by the interaction of the many individual agents' control laws should have the desired behavior as an attractor, preferably a global attractor. The performance of the system is judged by the global behavior of the system—it must be evaluated over all the agents. Example tasks include sensor coverage [3], [14], formation control [1], [5], [7], [9]–[13], [15], [17], multi-agent pursuer-evader, and other types of self-organization, including static and dynamic self-assembly.

The key constraints are that each agent may have significant dynamics and limited sensing, computation, motion, and communication capabilities. The behavior of the group should improve/degrade gracefully as agents are added or deleted; in other words, the approach should be scalable, robust, and require no central controller. Furthermore, unlike the work in [1], we do not assume that the agents have access to global swarm information.

While intelligent collective behavior can emerge even when each agent is ignorant of the global behavior of the system, the framework we are pursuing involves equipping each agent with (1) an estimator that allows it to estimate the relevant global performance measures for the swarm, and (2) a local controller that drives the performance measures to their desired values. The challenge is to design the individual estimators and controllers so that the induced global "emergent" behavior is the desired behavior.

In this paper we apply this approach to the problem of controlling the formation of a swarm of robots. Our primary interest is in large swarms where it is not necessary to specify the exact position of each robot. Instead, the swarm formation can be described by a set of summary statistics that form a basis for the space of all formations. These statistics should have the property that low-order statistics capture much of the essential shape of the swarm, but progressively higher-order statistics can be specified until only a single formation is consistent with the statistics. Low-order statistics then provide a convenient abstraction of the total swarm formation, allowing, for example, high-level human control of a large number of robots. The desired summary statistics can be optimized according to the changing tasks of the swarm.

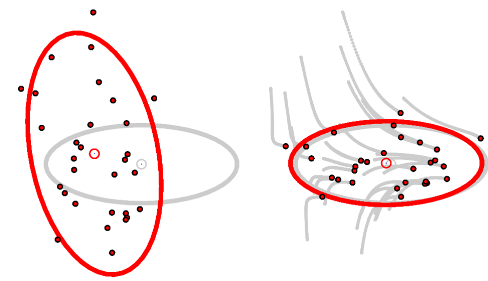

Fig. 1. (Left) The initial configuration of a swarm, a uniform-mass ellipse with the same first and second inertial moments as the swarm, and the goal formation of the swarm represented as another uniform-mass ellipse. (Right) The swarm converges to the desired formation statistics.

Formation inertial moments are an example of a class of summary statistics which fit our framework. Our simulations will focus on the first- and second-order inertial moments, the lowest-order moments providing information on the position, orientation, and "shape" of the swarm. We will describe controllers for the individual robots that drive these formation moments to desired values (Figure 1). Each robot can sense its own position and velocity, control its acceleration, and exchange information with nearby robots. As a result, the robots form a communication graph with changing topology as the robots move. Each robot implements an estimator that maintains an estimate of the current swarm formation statistics, based on its own sensed data and information received from neighbors, and a nonlinear motion control algorithm. The main results of this paper, Theorems 1 and 2, provide guarantees on the convergence of the swarm to the desired formation statistics for two different combinations of estimators and controllers.

II. INERTIAL MOMENT STATISTICS

Suppose we have n point robots (i.e., small relative to the size of the formation) in an m-dimensional space. Assuming that the "mass" of each robot in the swarm is identical, the position of robot i is $p_i = [p_{ix}\ p_{iy}\ p_{iz}]^T$ for $m = 3$, and

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normalizing by setting the total mass of the swarm to be unit, the inertial moments M_{abc} are

$$M_{abc} = \frac{1}{n}\sum_{i=1}^{n} p_{ix}^{a} p_{iy}^{b} p_{iz}^{c} \tag{1}$$

where $a, b, c \geqslant 0$ and where $a+b+c$ is called the *order* of the moment statistic. Given a particular robot formation, a sufficient number of moment statistics is guaranteed to distinguish it from any other formation. In other words, moments can provide an exact formation description. We are interested, however, in the case where a small number of low-order moments is used to specify a family of formations. If ℓ moment constraints are specified on n robots in an m-dimensional space, in general there is an $(mn-\ell)$-dimensional space of swarm configurations that satisfy the constraints. The structure and topology of such formation spaces can be studied using tools from real algebraic geometry [2].

In this paper we focus on formations defined by first- and second-order moments. The m first-order moments specify the center of mass of the swarm. From the $m(m+1)/2$ second-order moments we can derive $m(m-1)/2$ variables describing the orientation of orthogonal principal axes of inertia of the swarm and m shape variables summarizing the elongation of the swarm along the principal axes. Our abstraction of the swarm formation, then, is given by the $m(m+1)/2$ group variables describing the position and orientation of the principal axis frame in $SE(m)$ and the m shape variables describing the elongation of the swarm along these axes [1].

III. PROBLEM STATEMENT

Suppose the swarm consists of n agents with positions $p_1, \ldots, p_n \in \mathbb{R}^m$, which we write also as the combined vector $p = [p_1^T \cdots p_n^T]^T \in \mathbb{R}^{mn}$. Given a C^2 vector moment generating function $\phi : \mathbb{R}^m \to \mathbb{R}^\ell$, we define the swarm moment vector $f(p) \in \mathbb{R}^\ell$ as

$$f(p) = \frac{1}{n}\sum_{i=1}^{n} \phi(p_i)\,. \tag{2}$$

Each agent has knowledge of a desired moment vector for the swarm, represented as the vector $f^\star \in \mathrm{Im}(f)$. The primary goal of each agent is to move itself to an equilibrium position so that the final swarm configuration p satisfies $f(p) = f^\star$. Each agent measures its own position and velocity and controls its own acceleration. Furthermore, each agent can communicate with its neighbors, namely, agents i and j can communicate with each other whenever $p_i \leftrightarrow p_j$, where $\leftrightarrow$ is a fixed symmetric relation on $\mathbb{R}^m$. For example, we may have $p_i \leftrightarrow p_j$ if and only if $|p_i - p_j| \leqslant r$, where r represents a communication radius. Thus each configuration $p \in \mathbb{R}^{mn}$ defines the graph of an underlying communication network, and we let $\mathfrak{C} \subset \mathbb{R}^{mn}$ denote the set of all such configurations for which this graph is connected. As the agents move with time, the topology of this network can change, but we will perform our stability analysis below under the assumption that $p(t) \in \mathfrak{C}$, namely, that the network remains connected in forward time. For this reason we will assume $f^\star \in f(\mathfrak{C})$.

Our algorithms will guarantee that the system always converges to an equilibrium, but there will usually be "bad" equilibria for which $f(p) \neq f^\star$. However, we will show that all bad equilibria are unstable (in the sense of Lyapunov). Our approach is based on following the gradients $\nabla\Xi$ of a potential function $\Xi : \mathbb{R}^{mn} \to \mathbb{R}$ of the form

$$\Xi(p) = \big[f(p) - f^\star\big]^T \Gamma \big[f(p) - f^\star\big] \tag{3}$$

where $\Gamma \in \mathbb{R}^{\ell\times\ell}$ is a suitably chosen global gain matrix. Specifically, given $f^\star \in \mathrm{Im}(f)$ and $D \subset \mathbb{R}^{mn}$, we let $\mathcal{G}(f^\star, D)$ denote the cone of all symmetric positive-definite matrices Γ such that $\Xi(p)$ has no local minima in D other than the global minima where $f(p) = f^\star$. We assume that $\mathcal{G}(f^\star) = \mathcal{G}(f^\star, \mathbb{R}^{mn})$ is nonempty. In fact, one can show that for any m and any $n \geqslant m+1$, the function ϕ which generates all inertial moments (1) of orders one and two is such that we can calculate members of $\mathcal{G}(f^\star)$ for every $f^\star \in \mathrm{Im}(f)$. For example, this is the case when $m = 2$, $\ell = 5$, $n \geqslant 3$, and

$$\phi(a,b) = \begin{bmatrix} a & b & a^2 & ab & b^2 \end{bmatrix}^T. \tag{4}$$

For higher-order inertial moments, we have been able to show only that $\mathcal{G}(f^\star, D) \neq \varnothing$ for every $f^\star \in \mathrm{Im}(f)$ and every *bounded* set $D \subset \mathbb{R}^{mn}$, in which case we can proceed with a semiglobal rather than a global analysis. To ensure bounded signals in our closed-loop systems, we further assume that ϕ and thus also f and Ξ are proper (i.e., radially unbounded) functions. Finally, we let the set $C = \{p \in \mathbb{R}^{mn} : \nabla\Xi(p) = 0\}$ denote the set of critical points of Ξ and assume the following:

(P) Each $\bar{p} \in C$ has neighborhood N such that if $p \in N \cap C$ then $\Xi(p) = \Xi(\bar{p})$.

There are large classes of functions ϕ for which **(P)** holds; for example, **(P)** is always satisfied whenever ϕ is a locally semialgebraic (e.g., polynomial) function [2]. However, there do exist C^∞ choices for ϕ such that **(P)** fails.

IV. NONLINEAR GRADIENT CONTROL WITH HIGH-PASS ESTIMATORS

In this section we consider the following local control and estimation algorithm for agent i:

$$\begin{aligned} \ddot{p}_i = -B_i\dot{p}_i - \big[D\phi(p_i)\big]^T \Lambda_i \big[D\phi(p_i)\big]\dot{p}_i \\ + \big[D\phi(p_i)\big]^T \Gamma \big[f^\star - x_i\big] \end{aligned} \tag{5}$$

$$\dot{w}_i = -\gamma w_i - \sum_{j\neq i} a(p_i, p_j)\big[x_i - x_j\big] \tag{6}$$

$$x_i = w_i + \phi(p_i)\,. \tag{7}$$

Here (5) represents the control law and (6)–(7) represent a consensus estimator, where $x_i(t) \in \mathbb{R}^\ell$ is the agent's current estimate of $f(p)$ and $w_i(t) \in \mathbb{R}^\ell$ is the internal estimator state. In the control law (5), $D\phi(\cdot)$ denotes the $\ell\times m$ Jacobian matrix of ϕ, the matrices $B_i \in \mathbb{R}^{m\times m}$ and $\Lambda_i \in \mathbb{R}^{\ell\times\ell}$

are constant local damping matrices, and $\Gamma \in \mathcal{G}(f^\star)$. In the estimator dynamics (6), $\gamma \geqslant 0$ is an estimator "forgetting factor" and $a : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is a C^1 symmetric function[1] such that $\operatorname{supp}(a) \subset \operatorname{Graph}(\leftrightarrow)$. Thus to implement this control algorithm, each agent i must continually transmit its current values of p_i and x_i to its neighbors. We call the estimator (6)–(7) a high-pass estimator because if the estimator gains $a(p_i, p_j)$ were constant, then the resulting LTI system taking the inputs $\phi(p_i)$ to the outputs x_i would be a high-pass filter. This estimator is based on the one introduced in [16].

Let $\mathbf{1} \in \mathbb{R}^n$ denote the vector of n ones, and let $\operatorname{Orth}(\mathbf{1})$ denote the collection of $n \times (n-1)$ matrices S such that $S^T S = I$ and $S^T \mathbf{1} = 0$ (namely, the columns of S form an orthonormal basis for $\operatorname{span}\{\mathbf{1}\}^\perp$). Then by orthogonal decomposition we have

$$I = SS^T + \frac{\mathbf{1}\mathbf{1}^T}{n} \tag{8}$$

and thus $ASS^T A^T \leqslant AA^T$ for any n-column real matrix A (in particular we have $|AS|_F \leqslant |A|_F$ where $|\cdot|_F$ denotes the Frobenius norm). We define the *Laplacian* $L(p) \in \mathbb{R}^{n\times n}$ to be the symmetric matrix whose off-diagonal elements in row i, column j are equal to $-a(p_i, p_j)$ and whose diagonal elements are the negatives of the sums of the off-diagonal elements in the same row (so that $L(p)\mathbf{1} \equiv 0$). Moreover, fixing $S \in \operatorname{Orth}(\mathbf{1})$, we define the *reduced Laplacian* $L^\star(p)$ to be the symmetric matrix

$$L^\star(p) = S^T L(p) S, \tag{9}$$

and we note from (8) that $SL^\star(p) = L(p)S$. Furthermore, for $p \in \mathfrak{C}$ and for positive estimator weights $a(\cdot,\cdot)$ on the connected arcs, the smallest eigenvalue of the reduced Laplacian $L^\star(p)$ (called the *algebraic connectivity* of the underlying graph) will be strictly positive [4], [6]. Our primary assumption on the communication network is that this eigenvalue is bounded away from zero, namely, that

$$L^\star(p) \geqslant \varepsilon I \tag{10}$$

along trajectories in forward time for some constant $\varepsilon > 0$. In particular, (10) implies that $p(t) \in \mathfrak{C}$ for all $t \geqslant t_0$. Note that ε can be made large for a connected network by scaling up the estimator gain function $a(\cdot,\,\cdot)$. Also, for a connected network with equal unit weights, the value of ε is bounded from below by $2 - 2\cos(\pi/n)$ [4]; hence if the agents know an upper bound on n then they can calculate a lower bound on ε. Next, we assume that the small-gain condition

$$\Gamma^2 + I < \varepsilon\big(\Lambda_i + \Lambda_i^T\big) \tag{11}$$

holds for each i. Finally, we assume that $\gamma = 0$ and that each state w_i has the initial value $w_i(t_0) = 0$; the more general cases will be discussed below.

Theorem 1: Let ϕ and $f^\star$ be such that ϕ is C^2 and proper, $f^\star \in f(\mathfrak{C})$, $\mathcal{G}(f^\star) \neq \varnothing$, and property **(P)** holds. Choose $\Gamma \in \mathcal{G}(f^\star)$, fix $B_i + B_i^T > 0$ and $w_i(t_0) = 0$ for each i, and choose $a(\cdot,\cdot)$ to be C^1 and symmetric. Let (10) and (11) hold for some $\varepsilon > 0$, and fix $\gamma = 0$. Then each trajectory of the system (5)–(7) is bounded in forward time and converges to an equilibrium; moreover, every bad equilibrium is unstable.

As will become evident in the proof of this theorem in Section VII, the dynamics of the estimator (6)–(7) include a subsystem of the form $\dot\chi = -\gamma\chi$ which is uncontrollable from the inputs $\phi(p_i)$ but observable through the estimation errors $e_i = f(p) - x_i$. If $\gamma = 0$ and if the states $w_i(t_0)$ are not initialized to zero, then the constants χ will generate persistent nonzero constant offsets in the error variables e_i. These steady-state estimation errors will cause the swarm to converge to a formation with the wrong statistics. To avoid such errors, one would have to somehow globally simultaneously reinitialize these states to zero whenever agents leave the swarm (e.g., due to failure) or new agents join the swarm. Furthermore, if $\gamma = 0$ then any additive noise in the communication channels will pass through pure integrators $\dot\chi =$noise, resulting in random drift in the estimation errors. To alleviate these problems one could choose $\gamma > 0$; in this case any incorrect initialization of the states $w_i(t_0)$ will be asymptotically forgotten, and communication noise will not cause random drift. However, the estimator (6)–(7) exhibits steady-state error under constant inputs, an error whose size is proportional to $\gamma/(\gamma+\varepsilon)$ (and hence nonzero for $\gamma > 0$). Nevertheless, as we will illustrate in Section VI, a small error due to a small positive γ may be preferable to errors caused by incorrect initializations. In the next section we introduce a more complex estimator which achieves the same advantages of choosing $\gamma > 0$ in (6) but does so without introducing any steady-state error.

The conclusion of Theorem 1 (and likewise of Theorem 2 below) remains valid if the damping matrices B_i are C^1 functions of the states p, $\dot p$, $x_1, \ldots, x_n$, and $w_1, \ldots, w_n$, provided $B_i(\cdot) + B_i^T(\cdot) > 0$ holds globally for each i (however, keep in mind that B_i cannot depend on information not available from the neighbors of agent i). Hence we can view these damping matrices B_i as additional sources of control, and we might design them to help maintain network connectivity or to help avoid collisions between agents. This extension is a topic of future research.

V. NONLINEAR GRADIENT CONTROL WITH PROPORTIONAL-INTEGRAL ESTIMATORS

In this section we assume that there exists a proper metric d on $\mathbb{R}^m$ such that

$$\sup_{p\in\mathfrak{C}} \max_{1\leqslant i,j\leqslant n} d(p_i, p_j) < \infty \tag{12}$$

(this holds in particular when $p_i \leftrightarrow p_j$ only if $d(p_i, p_j) \leqslant r$, where $r > 0$ is a fixed communication radius). It follows that there exists a C^1 function $\zeta : \mathbb{R}^m \to \mathbb{R}$ such that

$$|\phi(p_i) - \phi(p_j)|^2 \leqslant \zeta(p_i) \tag{13}$$

[1] If $\mathcal{X}$ and $\mathcal{Y}$ are nonempty sets, we say that a function $\psi : \mathcal{X}\times\mathcal{X} \to \mathcal{Y}$ is *symmetric* when $\psi(a,b) = \psi(b,a)$ for all $a, b \in \mathcal{X}$.

for all $p \in \mathfrak{C}$ and any $i, j \in \{1, \ldots, n\}$ [8, Corollary A.15]. Let $a, b : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be bounded C^1 symmetric functions such that $\mathrm{supp}(a) \cup \mathrm{supp}(b) \subset \mathrm{Graph}(\leftrightarrow)$. We also assume that b has bounded first-order partial derivatives. Consider the following local control/estimation algorithm:

$$\begin{aligned} \ddot{p}_i = &- B_i \dot{p}_i - \big[D\phi(p_i)\big]^T \Lambda_i \big[D\phi(p_i)\big] \dot{p}_i \\ &- c_i \zeta(p_i) \dot{p}_i + \big[D\phi(p_i)\big]^T \Gamma \big[f^\star - x_i\big] \end{aligned} \tag{14}$$

$$\begin{aligned} \dot{x}_i = &- \gamma x_i - \sum_{j \neq i} a(p_i, p_j) \big[x_i - x_j\big] \\ &+ \sum_{j \neq i} b(p_i, p_j) \big[w_i - w_j\big] + \gamma \phi(p_i) \end{aligned} \tag{15}$$

$$\dot{w}_i = - \sum_{j \neq i} b(p_i, p_j) \big[x_i - x_j\big] . \tag{16}$$

As before, $\gamma > 0$ is a global forgetting factor (but with $\gamma = 0$ no longer allowed), $B_i \in \mathbb{R}^{m \times m}$ and $\Lambda_i \in \mathbb{R}^{\ell \times \ell}$ are constant local damping matrices, $c_i > 0$ is a local damping parameter, and $\Gamma \in \mathcal{G}(f^\star)$. To implement this controller, each agent i must continually transmit its current values of p_i, w_i, and x_i to its neighbors. When compared with (5), the control law (14) has an additional damping term $-c_i \zeta(p_i) \dot{p}_i$ which will be useful for proving global stability. Furthermore, the estimator (15)–(16) has twice as many states as the one in (6)–(7), but it is no longer high-pass as there is no direct feedthrough from $\phi(p_i)$ to x_i.

We define the *proportional Laplacian* $L_P(p) \in \mathbb{R}^{n \times n}$ to be the symmetric matrix whose off-diagonal elements in row i, column j are equal to $-a(p_i, p_j)$ and whose diagonal elements are such that $L_P(p)\mathbf{1} \equiv 0$. We define the *integral Laplacian* $L_I(p) \in \mathbb{R}^{n \times n}$ in the same way but using $b(\cdot, \cdot)$ instead of $a(\cdot, \cdot)$. Again fixing $S \in \mathrm{Orth}(\mathbf{1})$, we define the corresponding reduced Laplacians $L_P^\star(p) = S^{\mathsf{T}} L_P(p) S$ and $L_I^\star(p) = S^{\mathsf{T}} L_I(p) S$. Our primary assumption on the communication network is that there exist constants $\rho > -\gamma$ and $\varepsilon > 0$ such that

$$\rho I \leqslant L_P^\star(p) \leqslant \bar{\rho} I \tag{17}$$

$$\varepsilon I \leqslant L_I^\star(p) \leqslant \bar{\varepsilon} I \tag{18}$$

along trajectories in forward time (again implying a connected network $p(t) \in \mathfrak{C}$). Here the constants $\bar{\rho}, \bar{\varepsilon} > 0$ represent upper bounds on the reduced Laplacians which exist because the functions a and b are bounded. Finally, we assume that a small-gain condition similar to (11) is satisfied (see (73) below), which we accomplish by choosing the damping parameters Λ_i and c_i sufficiently large.

Theorem 2: Let ϕ and $f^\star$ be such that ϕ is C^2 and proper, $f^\star \in f(\mathfrak{C})$, $\mathcal{G}(f^\star) \neq \varnothing$, and property **(P)** holds. Assume (12) holds, choose $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ to be C^1, bounded, symmetric, and such that b has bounded first-order partial derivatives. Choose $\Gamma \in \mathcal{G}(f^\star)$, fix $B_i + B_i^T > 0$, $\Lambda_i + \Lambda_i^T > 0$, and $c_i > 0$ for each i, and assume (17) and (18) hold for some $\rho > -\gamma$ and $\varepsilon > 0$ (with $\gamma > 0$). If the damping parameters Λ_i and c_i are sufficiently large, then each trajectory of the system (14)–(16) is bounded in forward time and converges to an equilibrium; moreover, every bad equilibrium is unstable.

Like the high-pass estimator (6)–(7) with $\gamma = 0$, the PI estimator (15)–(16) includes a subsystem of the form $\dot{\chi} = 0$ which is uncontrollable from the inputs $\phi(p_i)$ (see the proof in Section VIII). Thus as before, χ might be nonzero due to inconsistent initializations and might drift due to communication noise. However, unlike the high-pass case, these states χ are not observable through the estimation errors $e_i = f(p) - x_i$, which means their behavior will not affect the swarm dynamics.

VI. SIMULATION RESULTS

We simulated the algorithms in Sections IV and V for a swarm of $n = 5$ planar robots ($m = 2$), ϕ as in (4), and $f^\star = [0 \quad 0 \quad 50 \quad 0 \quad 50]^T$. The controller gain matrix was $\Gamma = \mathrm{diag}(80, 80, 8, 8, 8)$. The estimator gain functions where chosen according to an equal weighting scheme with a communication radius of 20: $a(p_i, p_j) = a_0$ and $b(p_i, p_j) = b_0$ when $|p_i - p_j| \leq 20$ and $a(p_i, p_j) = b(p_i, p_j) = 0$ otherwise (the fact that these gain functions are discontinuous had little effect on the simulations). Also, we set the nonlinear damping gains Λ_i and c_i in (5) and (14) to zero as the constants B_i can provide adequate damping over a bounded region.

We first simulated the high-pass scheme of Section IV with damping $B_i = 40I$, estimator gain $a_0 = 5$, and no forgetting factor ($\gamma = 0$). Figure 2 shows the results of the inertial moments M_{10} =CMx (the first component of f) and M_{20} =Iyy (the third component of f). The first 15 seconds show the convergence of the formation statistics to their desired values with no steady-state error. At time $t = 15$, one of the agents fails and leaves the swarm, resulting in a permanent nonzero steady-state error after that point. Actually, the remaining agents do not move at all from their equilibria after time $t = 15$, demonstrating that the high-pass estimator with no forgetting factor does not recover from initialization errors. If we include a forgetting factor of $\gamma = 0.7$, then we do recover from the loss of the agent (Figure 3), but we now incur a small nonzero steady-state error both before and after the loss.

We next simulated the PI scheme of Section V with increased damping $B_i = 100I$ (to account for the additional neglected nonlinear damping terms), estimator gains $a_0 = 2$ and $b_0 = 0.02$, and $\gamma = 6$. Figure 4 shows that the PI algorithm can also recover from the loss of an agent (again at time $t = 15$) but now with zero steady-state error.

VII. PROOF OF THEOREM 1

Defining $z_i, e_i \in \mathbb{R}^\ell$ as

$$z_i = \frac{d}{dt} \phi(p_i) = \big[D\phi(p_i)\big] \dot{p}_i \tag{19}$$

$$e_i = f(p) - x_i \,, \tag{20}$$

we introduce the following $\ell \times n$ matrices:

$$X = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \tag{21}$$

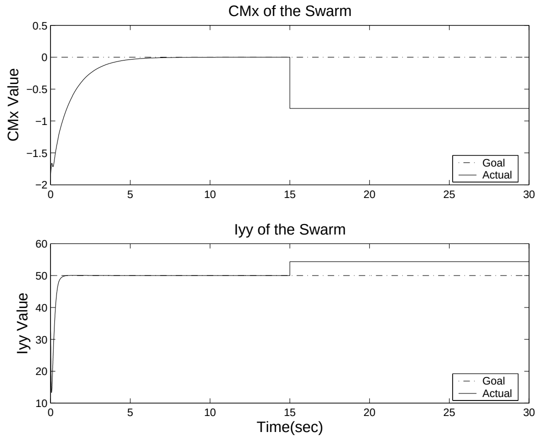

Fig. 2. The high-pass algorithm with no forgetting factor ($\gamma = 0$).

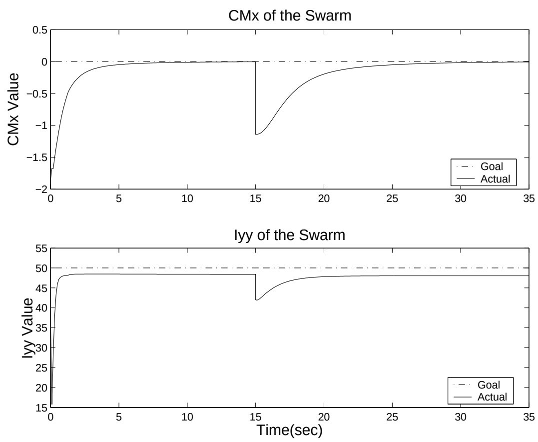

Fig. 3. The high-pass algorithm with forgetting factor $\gamma = 0.7$.

$$W = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} \tag{22}$$
$$\Phi(p) = \begin{bmatrix} \phi(p_1) & \cdots & \phi(p_n) \end{bmatrix} \tag{23}$$
$$E = \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix} = \Phi(p)\,\frac{\mathbf{1}\mathbf{1}^T}{n} - X \tag{24}$$
$$Z = \begin{bmatrix} z_1 & \cdots & z_n \end{bmatrix} = \frac{d}{dt}\,\Phi(p)\,. \tag{25}$$

Hence we may write the collection of consensus estimators (6)–(7) in matrix form as

$$\dot{W} = -\gamma W - XL(p) \tag{26}$$
$$X = W + \Phi(p)\,. \tag{27}$$

We write the complete state of the closed-loop system as either the triple $(p, \dot{p}, W)$, or with the global coordinate change given by (24) and (27), the triple $(p, \dot{p}, E)$. We see from (5) that the derivative of the Lyapunov function

$$V(p, \dot{p}) = \dot{p}^T\dot{p} + n\,\Xi(p) \tag{28}$$

can be written as

$$\dot{V} = \sum_{i=1}^{n}\Big[-\dot{p}_i^T\big[B_i + B_i^T\big]\dot{p}_i - z_i^T\big[\Lambda_i + \Lambda_i^T\big]z_i + 2z_i^T\Gamma e_i\Big]\,. \tag{29}$$

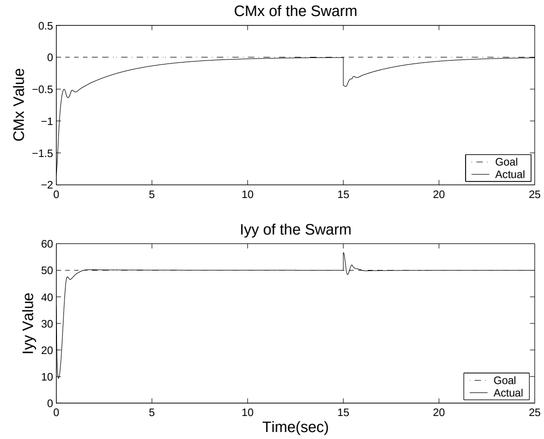

Fig. 4. The PI algorithm.

We observe from (24) and (27) that $E\mathbf{1} \equiv -W\mathbf{1}$ and thus

$$\dot{E}\mathbf{1} = \gamma W\mathbf{1} = 0 \tag{30}$$

which means

$$E(t)\mathbf{1} = -W(t_0)\mathbf{1} = 0 \tag{31}$$

for any $t \geqslant t_0$. Because $ES \equiv -XS$, we can write

$$\dot{E}S = XL(p)S - ZS = -ES \cdot L^\star(p) - ZS\,. \tag{32}$$

Also because $E\mathbf{1} \equiv 0$ we have $EE^T \equiv ESS^TE^T$, so

$$\frac{d}{dt}\,EE^T \leqslant -\varepsilon EE^T + \frac{1}{\varepsilon}ZZ^T\,. \tag{33}$$

Defining the Lyapunov function $U = \mathrm{Tr}(EE^T)$ we see that

$$\dot{U} \leqslant -\varepsilon U + \frac{1}{\varepsilon}\,\mathrm{Tr}(ZZ^T) = \sum_{i=1}^{n}\Big[-\varepsilon|e_i|^2 + \frac{1}{\varepsilon}|z_i|^2\Big]\,. \tag{34}$$

Furthermore, we can bound (29) from above as

$$\dot{V} \leqslant \sum_{i=1}^{n}\Big[-\dot{p}_i^T\big[B_i + B_i^T\big]\dot{p}_i - z_i^T\big[\Lambda_i + \Lambda_i^T - \frac{1}{\varepsilon}\Gamma^2\big]z_i + \varepsilon|e_i|^2\Big]\,. \tag{35}$$

To combine the Lyapunov functions V and U, we first use (11) to choose $\mu > 0$ such that

$$\Gamma^2 + (1+\mu)I \leqslant \varepsilon\big(\Lambda_i + \Lambda_i^T\big) \tag{36}$$

for each i. We then define $\Upsilon(p, \dot{p}, E) = V + (1+\mu)U$ and use (34), (35), and (36) to obtain

$$\dot{\Upsilon} \leqslant \sum_{i=1}^{n}\Big[-\dot{p}_i^T\big[B_i + B_i^T\big]\dot{p}_i - \mu\varepsilon|e_i|^2\Big]\,. \tag{37}$$

In particular, $\Upsilon(t)$ is nonincreasing along trajectories in forward time. Because $\Xi(p)$ is proper, Υ is a proper function of the states p, $\dot{p}$, and E, and we can conclude that all signals are bounded in forward time. By LaSalle's theorem we further conclude that every trajectory converges to an equilibrium at which $\dot{p} = 0$ and $E = 0$.[2] Thus the equilibria

[2]Actually, LaSalle's theorem only guarantees convergence to an equilibrium set; further arguments (omitted due to space constraints) are required to show convergence to a single equilibrium.

are at those values of p for which

$$\big[D\phi(p_i)\big]^T \Gamma \big[f^\star - f(p)\big] = 0 \tag{38}$$

for every i, or equivalently where $\nabla\Xi(p) = 0$. Recall that bad equilibria are those for which $f(p) \neq f^\star$; we now show that every bad equilibrium is unstable. Suppose $\bar{p} \in \mathbb{R}^{mn}$ is such that the point $(p,\dot{p},E) = (\bar{p},0,0)$ is a bad equilibrium. It follows from property **(P)** that there exists a neighborhood N of $\bar{p}$ such that if $p \in N$ satisfies $\nabla\Xi(p) = 0$ then $\Xi(p) = \Xi(\bar{p})$. By assumption $\bar{p}$ is not a local minimum of Ξ, which means there are points p_0 arbitrarily close to $\bar{p}$ for which $\Xi(p_0) < \Xi(\bar{p})$ and therefore also $\Upsilon(p_0,0,0) < \Upsilon(\bar{p},0,0)$. Now any trajectory starting from such a state $(p_0,0,0)$ converges to an equilibrium state $(p_1,0,0)$ for which $\nabla\Xi(p_1) = 0$, and because Υ is nonincreasing along trajectories we also have $\Xi(p_1) < \Xi(\bar{p})$. It follows that $p_1 \notin N$, namely, that this trajectory leaves the N-tube around the point $(\bar{p},0,0)$, and we conclude that this bad equilibrium is unstable.

VIII. PROOF OF THEOREM 2

The derivative of the Lyapunov function (28) is

$$\dot{V} = \sum_{i=1}^{n} \Big[-\dot{p}_i^T \big[B_i + B_i^T\big]\dot{p}_i - 2c_i\zeta(p_i)|\dot{p}_i|^2 - z_i^T\big[\Lambda_i + \Lambda_i^T\big]z_i + 2z_i^T\Gamma e_i\Big] \tag{39}$$

with z_i and e_i as in (19)–(20). We may write the collection of PI estimators (15)–(16) in matrix form as

$$\dot{X} = -X\big[\gamma I + L_P(p)\big] + WL_I(p) + \gamma\Phi(p) \tag{40}$$

$$\dot{W} = -XL_I(p) \tag{41}$$

with X, W, and Φ from (21)–(23). Defining E and Z as in (24)–(25) we obtain

$$\dot{E}\mathbf{1} = -\gamma E\mathbf{1} + Z\mathbf{1}\,, \qquad \dot{W}\mathbf{1} = 0\,. \tag{42}$$

From (8) we have $L_P \equiv L_P SS^T$ and $L_I \equiv L_I SS^T$, which means we can multiply both sides of (40)–(41) from the right by S to obtain

$$\dot{X}S = -XS\big[\gamma I + L_P^\star(p)\big] + WS\cdot L_I^\star(p) + \gamma\Phi(p)S \tag{43}$$

$$\dot{W}S = -XS\cdot L_I^\star(p)\,. \tag{44}$$

With the change of variables

$$Y = WS + \gamma\Phi(p)S\big[L_I^\star(p)\big]^{-1} \tag{45}$$

$$\Omega = \begin{bmatrix} XS & Y \end{bmatrix} \tag{46}$$

the equations (43)–(44) become

$$\dot{\Omega} = \Omega F^T + NH^T\,, \tag{47}$$

where

$$F = \begin{bmatrix} -\gamma I - L_P^\star(p) & L_I^\star(p) \\ -L_I^\star(p) & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{48}$$

$$N = \gamma ZS\big[L_I^\star(p)\big]^{-1} + \gamma\Phi(p)S\frac{d}{dt}\big[L_I^\star(p)\big]^{-1}. \tag{49}$$

We will write N as the sum

$$N = \gamma\sum_{i=0}^{n} N_i \tag{50}$$

where

$$N_0 = ZS\big[L_I^\star(p)\big]^{-1} \tag{51}$$

$$N_i = -\Phi(p)S\big[L_I^\star(p)\big]^{-1}\sum_{k=1}^{m}\frac{\partial L_I^\star(p)}{\partial p_i(k)}\big[L_I^\star(p)\big]^{-1}\dot{p}_i(k) \quad \text{for } 1 \leqslant i \leqslant n \tag{52}$$

and $p_i = [p_i(1)\,\cdots\,p_i(m)]^T \in \mathbb{R}^m$. We now derive bounds on these matrices N_i. First, using (18) we obtain

$$N_0N_0^T = ZS\big[L_I^\star(p)\big]^{-2}S^TZ^T \leqslant \frac{1}{\varepsilon^2}ZSS^TZ^T \tag{53}$$

Next, using (13) and our assumption that $p(t) \in \mathfrak{C}$ for all $t \geqslant t_0$, we obtain

$$|\Phi(p)S|_F^2 = \Big|\big[\Phi(p) - \phi(p_i)\mathbf{1}^T\big]S\Big|_F^2 \leqslant \big|\Phi(p) - \phi(p_i)\mathbf{1}^T\big|_F^2 \leqslant \sum_{j\neq i}|\phi(p_j) - \phi(p_i)|^2 \leqslant (n-1)\zeta(p_i)\,. \tag{54}$$

It follows from (18) and the fact that b has bounded partial derivatives that there exist constants $k_i > 0$ such that

$$|N_i|_F^2 \leqslant k_i\zeta(p_i)|\dot{p}_i|^2 \tag{55}$$

for $1 \leqslant i \leqslant n$. Each k_i depends on n, ε, and the bounds on the partial derivatives of b. Let σ be a constant such that $0 < \sigma < 1$ and

$$\sigma \leqslant \frac{\varepsilon(\gamma+\rho)}{(\gamma+\bar{\rho}^2) + 2\varepsilon\bar{\varepsilon}}\,. \tag{56}$$

Then the positive definite matrices

$$P = \begin{bmatrix} I & -\sigma I \\ -\sigma I & I \end{bmatrix}, \quad Q = \begin{bmatrix} (\gamma+\rho)I & 0 \\ 0 & \sigma\varepsilon I \end{bmatrix} \tag{57}$$

satisfy

$$(1-\sigma)I \leqslant P \leqslant (1+\sigma)I \tag{58}$$

and

$$\begin{aligned} &PF + F^TP + Q = \\ &\begin{bmatrix} -2L_P^\star(p) + (\rho-\gamma)I + 2\sigma L_I^\star(p) & \sigma\gamma I + \sigma L_P^\star(p) \\ \sigma\gamma I + \sigma L_P^\star(p) & -2\sigma L_I^\star(p) + \sigma\varepsilon I \end{bmatrix} \\ &\leqslant -\sigma\cdot\underbrace{\begin{bmatrix} \big(\frac{1}{\sigma}(\gamma+\rho) - 2\bar{\varepsilon}\big)I & -\gamma I - L_P^\star(p) \\ -\gamma I - L_P^\star(p) & \varepsilon I \end{bmatrix}}_{R(p)} \leqslant 0 \end{aligned} \tag{59}$$

because (56) implies that $R(p) \geqslant 0$. Let $\kappa > 0$ be such that

$$\kappa < \min\Big\{\frac{\gamma+\rho}{\sigma},\, \sigma\varepsilon\Big\}\,. \tag{60}$$

Then we have

$$PHH^TP = P - \begin{bmatrix} (1-\sigma^2)I & 0 \\ 0 & 0 \end{bmatrix} \tag{61}$$

and thus also

$$\begin{aligned} Q - &\frac{\kappa}{1+\sigma}PHH^TP \\ &= \begin{bmatrix} \big[\gamma+\rho+\kappa(1-\sigma)\big]I & 0 \\ 0 & \sigma\varepsilon I \end{bmatrix} - \frac{\kappa}{1+\sigma}P \\ &\geqslant \min\big\{\gamma+\rho+\kappa(1-\sigma),\, \sigma\varepsilon\big\}I - \kappa I = \alpha I\,, \end{aligned} \tag{62}$$

where $\alpha = \min\{\gamma + \rho - \sigma\kappa,\, \sigma\varepsilon - \kappa\}$. It now follows that

$$PF + F^T P + \frac{\kappa}{1+\sigma} PHH^T P \leqslant -\alpha I\,. \tag{63}$$

We define the matrix

$$\Psi = \Omega P \Omega^T + \beta E \mathbf{1}\mathbf{1}^T E^T + W \mathbf{1}\mathbf{1}^T W^T \tag{64}$$

where $\beta > 0$ is a constant parameter. Defining

$$\xi = \frac{\gamma^2 (n+1)(\sigma+1)}{\kappa}\,, \tag{65}$$

we use (8), (42), (47), (50), (53), and (63) to obtain

$$\begin{aligned}\dot{\Psi} &= \Omega\big[PF + F^T P\big]\Omega^T + \gamma \sum_{i=0}^{n} \big[N_i H^T P \Omega^T + \Omega P H N_i^T\big] \\ &\quad - 2\beta\gamma E\mathbf{1}\mathbf{1}^T E^T + \beta Z \mathbf{1}\mathbf{1}^T E^T + \beta E \mathbf{1}\mathbf{1}^T Z^T \qquad (66)\\ &\leqslant -\alpha \Omega\Omega^T - \beta\gamma E \mathbf{1}\mathbf{1}^T E^T \\ &\quad + \max\Big\{\frac{n\beta}{\gamma}, \frac{\xi}{\varepsilon^2}\Big\}\, ZZ^T + \xi \sum_{i=1}^{n} N_i N_i^T\,. \qquad (67)\end{aligned}$$

Because $ES = -XS$ we also have

$$\Omega\Omega^T = XSS^T X^T + YY^T = ESS^T E^T + YY^T \tag{68}$$

and therefore

$$\dot{\Psi} \leqslant -\nu_1 EE^T - \alpha YY^T + \nu_2\, ZZ^T + \xi \sum_{i=1}^{n} N_i N_i^T\,, \tag{69}$$

where

$$\nu_1 = \min\big\{\alpha,\, n\beta\gamma\big\} \tag{70}$$

$$\nu_2 = \max\Big\{\frac{n\beta}{\gamma}, \frac{\xi}{\varepsilon^2}\Big\}\,. \tag{71}$$

Defining the Lyapunov function $U = \mathrm{Tr}(\Psi)$ we see that

$$\dot{U} \leqslant -\alpha |Y|_F^2 + \sum_{i=1}^{n} \Big[-\nu_1 |e_i|^2 + \nu_2 |z_i|^2 + \xi k_i \zeta(p_i) |\dot{p}_i|^2\Big]\,.$$

Furthermore, we can bound (39) from above as

$$\begin{aligned}\dot{V} = \sum_{i=1}^{n} \Big[&-\dot{p}_i^T \big[B_i + B_i^T\big]\dot{p}_i - 2c_i\zeta(p_i)|\dot{p}_i|^2 \\ &- z_i^T\big[\Lambda_i + \Lambda_i^T - \frac{1}{\nu_1}\Gamma^2\big] z_i + \nu_1 |e_i|^2\Big]\,. \qquad (72)\end{aligned}$$

Finally, we assume that

$$\Gamma^2 + \nu_1\nu_2 I < \nu_1\big(\Lambda_i + \Lambda_i^T\big) \quad \text{and} \quad \xi k_i < 2c_i \tag{73}$$

for each i. To combine the Lyapunov functions V and U, we first use (73) to choose $\mu > 0$ such that

$$\Gamma^2 + \nu_1\nu_2(1+\mu) I \leqslant \nu_1\big(\Lambda_i + \Lambda_i^T\big) \tag{74}$$

$$\xi k_i (1+\mu) \leqslant 2c_i \tag{75}$$

for each i. Thus $\Upsilon(p, \dot{p}, E, W) = V + (1+\mu)U$ satisfies

$$\dot{\Upsilon} \leqslant -\alpha |Y|_F^2 + \sum_{i=1}^{n} \Big[-\dot{p}_i^T\big[B_i + B_i^T\big]\dot{p}_i - \mu\nu_1 |e_i|^2\Big]\,. \tag{76}$$

Using arguments similar to the ones used in Section VII, we may conclude that all signals are bounded in forward time, that every trajectory converges to an equilibrium for which $\nabla\Xi(p) = 0$, and that every bad equilibrium is unstable.

IX. Future Work

The controller extends trivially to the case in which the robots are kinematic and holonomic, but more work is needed for the nonholonomic and underactuated cases. Also, although we expect the PI estimator to have better sensor noise attenuation properties than the high-pass estimator, further analysis is needed. Similarly, we need to investigate the effects of noise and time delay in the communication channels between the robots. We may also consider adaptive algorithms for adjusting the estimator gains. Finally, instead of the regulation problem, we could consider the tracking problem in which the desired formation moment vector $f^\star$ changes with time.

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