C91 FUNDAMENTALS OF CONTROL SYSTEMS
Using Routh-Hurwitz

1. General Procedure

The Routh-Hurwitz (RH) Criterion is a general mathematical technique that may be used to determine how many of the roots of a characteristic equation such as the one below have positive real parts, and are therefore unstable[1]. The general form for a characteristic equation is:

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

We'll assume that all of the a_i's are of the same sign (usually taken to be positive) — if they aren't then there is definitely at least one unstable root. The first step in the RH test is to construct the Routh Array, as follows:

$$\begin{array}{c|lllll}
s^n & a_n & a_{n-2} & a_{n-4} & \cdots & a_0 \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots & 0 \\
s^{n-2} & b_{n-2} & b_{n-4} & \cdots & & \\
s^{n-3} & c_{n-3} & c_{n-5} & \cdots & & \\
\vdots & & & & & \\
s^0 & \bullet_0 & & & &
\end{array}$$

The array has $n+1$ rows. It is formed by first filling in the s^n and s^{n-1} rows with the coefficients from the characteristic equation, as shown. In the illustration above, it is assumed that n is even. If n were odd, then a_0 would appear in the s^{n-1} row. Having done this, it is time to fill in the s^{n-2} row. The appropriate coefficients are found as follows:

$$b_{n-2} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_{n-4} = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$b_{n-i} = \frac{a_{n-1}a_{n-i} - a_n a_{n-i-1}}{a_{n-1}}$$

Notice the pattern — it is a lot like finding 2×2 determinants. You keep on filling in this row until i is greater than or equal to n, where $i = 2,4,6,\ldots$ The s^{n-3} row can now be found in an exactly analogous manner from the s^{n-1} and s^{n-2} rows. The coefficients are given by:

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[1]Of course, a root is just a number, neither stable or unstable. However, since poles that have positive real parts correspond to unstable behavior, I will use the control engineer's vernacular: "unstable pole" or "unstable root".

$$c_{n-3} = \frac{b_{n-2}a_{n-3} - a_{n-1}b_{n-4}}{b_{n-2}}$$

$$c_{n-5} = \frac{b_{n-2}a_{n-5} - a_{n-1}b_{n-6}}{b_{n-2}}$$

$$c_{n-i} = \frac{b_{n-2}a_{n-i} - a_{n-1}b_{n-i-1}}{b_{n-2}}$$

Again, this row is filled in until $i \geq n$, where $i = 3,5,7,...$ Row after row can be filled in in an analogous manner until the s^0 row is reached. This row will have only one entry.

Having completed the array, you may use it to determine the number of unstable roots. The basic rule is:

> **The number of poles in the right half plane equals the number of sign changes in the first column of the Routh Array.**

The first column is, of course:

$$\begin{array}{c} a_n \\ a_{n-1} \\ b_{n-2} \\ c_{n-3} \\ \vdots \\ \bullet_0 \end{array}$$

A sign change occurs when two values, one immediately above the other, do not have the same sign; e.g., $\text{sign}(a_{n-1}) \neq \text{sign}(b_{n-2})$.

1.1 Example

Suppose that the characteristic equation is:

$$(s + 1)(s + 2)(s - 1 + 2j)(s - 1 - 2j) = s^4 + s^3 + s^2 + 11s + 10 = 0$$

Because I have shown this in factored form, we know that two of the roots are unstable. Pretend that you did not know this. You can find out from the Routh Array:

s^4	1	1	10
s^3	1	11	0
s^2	$-10 = (1 \cdot 1 - 1 \cdot 11) / 1$	$10 = (1 \cdot 10 - 1 \cdot 0) / 1$	0
s^1	$12 = (-10 \cdot 11 - 1 \cdot 10) / (-10)$	0	

s^0	$10 = (12 \cdot 10 + 10 \cdot 0) / 12$

The first column is 1, 1, -10, 12, 10. It has two sign changes (1 to -10, and -10 to 12), and therefore indicates two unstable roots.

2. Zeros in the First Column

How do you interpret a zero in the first column? Is there a sign change or isn't there, or does it just mean that some roots lie on the imaginary axis (the borderline stability case)? Toughest of all, how do you evaluate the rest of the array after a zero has appeared in the first column — division by zero will be required. The way to deal with this occurrence is to replace the zero with a very small positive number, ε, and to complete the array. For instance, if the characteristic equation is:

$$(s + 2)(s + j)(s - j) = s^3 + 2s^2 + s + 2 = 0$$

Then the array is:

s^3	1	1
s^2	2	2
s^1	$0 \approx \varepsilon$	
s^0	$2 = (\varepsilon \cdot 2 - 2 \cdot 0) / \varepsilon$	

The first column is 1, 2, 0, 2. As this example illustrates, *when the sign of the entry above the zero is the same as that below it, the zero indicates that there is a pair of imaginary roots.*

It is not always the case, however, that the signs are the same above and below. For instance, consider the characteristic equation:

$$(s + .973 + .787j)(s + .973 - .787j)(s - .473 + 1.025j)(s - .473 - 1.025j) = s^4 + s^3 + s^2 + s + 2 = 0$$

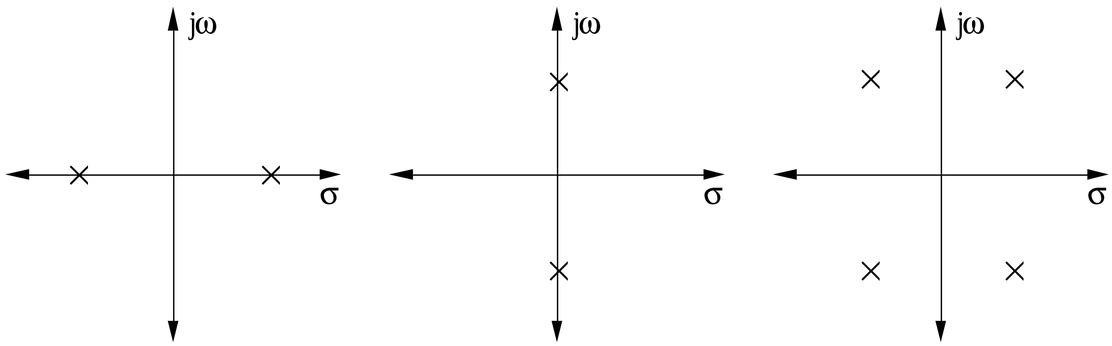
The array is:

s^4	1	1	2
s^3	1	1	0
s^2	$0 \approx \varepsilon$	2	
s^1	$(\varepsilon - 2) / \varepsilon$		
s^0	2		

For a small positive ε, the signs in the first column are +, +, +, -, +, and we interpret this as two sign changes. In general, *when the sign of the entry* *above the zero (ε) is different from that below it, this constitutes a single sign change.*

3. A Row of Zeros

Another pathological case occurs when an entire row of the Routh Array contains zeros. This will be caused by poles that are located in symmetric patterns about the origin, as in any of the illustrations below.



For instance, suppose the characteristic equation is:

$$(s + 2j)(s - 2j)(s + 3j)(s - 3j) = s^4 + 13s^2 + 36 = 0$$

Then the array starts out as:

s^4	1	13	36
s^3	0	0	0

When an entire row is composed of zeros like this, it is not sufficient to replace the first one by an ε. Instead, we must make use of what is known as the *auxiliary polynomial*. When the s^i row is zero, the auxiliary polynomial is formed from the coefficients of the s^{i+1} row. In this case, the auxiliary polynomial is:

$$P(s) = s^4 + 13s^2 + 36$$

The trick to completing the array is to replace the s^i row by the coefficients of dP/ds:

$$\frac{dP(s)}{ds} = 4s^3 + 26s$$

The new array is:

s^4	1	13	36
s^3	4	26	0
s^2	6.5	36	
s^1	3.85	0	
s^0	36		

This result correctly indicates that there are no unstable roots; however, it does not give us any information regarding imaginary roots.

3.1 Example of the Use of Auxiliary Polynomials

Consider the following characteristic equation:

$$s^5 + 2s^4 + 4s^3 + 8s^2 + 5s + 10 = 0$$

The array starts out as:

s^5	1	4	5
s^4	2	8	10
s^3	0	0	

The s^3 row is all zeros, so we form an auxiliary polynomial from the s^4 row:

$$P(s) = 2s^4 + 8s^2 + 10$$

$$\frac{dP}{ds} = 8s^3 + 16s$$

The coefficients of the s^3 row are replaced by those of dP/ds, and we can complete the array:

s^5	1	4	5
s^4	2	8	10
s^3	8	16	
s^2	4	10	
s^1	-4	0	
s^0	10		

There are two sign changes in the first row, indicating two unstable roots. In fact, the roots are $.3436 \pm 1.4553j$, $-.3436 \pm 1.4553j$, and -2.

It is interesting to note that the symmetric roots of the auxiliary polynomial *are* roots of the characteristic equation; thus, we can often find these roots analytically.

4. A Few Comments

The Routh-Hurwitz Criterion is getting to be a bit antiquated. You may wonder why bother with it at all when a program like MATLAB can find roots for you in next to no time. There are two reasons I can give for this. First, remember that for a closed loop system the roots are functions of the controller parameters, such as the gain (K). It is sometimes useful to have an analytical means of solving for the maximum value of K that you can use before getting unstable poles. Second, you may sometimes want to do a complete analysis symbolically (without numbers), for instance to find relationships between plant design parameters and controller parameters that give the best performance. Here again, Routh-Hurwitz is useful.

As a final note, the RH Criterion can also be used to perform a *relative stability analysis*, which will tell you how many of the roots have real parts that are greater than any value you select. For instance, rather than being concerned with stability, you may want to know how may poles have exponential decays slower than $\tau = 0.5$ sec. To do this, you simply substitute $s = z - 2$ into the characteristic equation, and apply the criterion to the new polynomial in z. The number of sign changes in the first column of the array tells you how many poles lie to the right of the vertical line $s = -2$.