A General Stability Condition for Multi-Agent Coordination by Coupled Estimation and Control

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***Abstract*— Many multi-agent coordinated behaviors can be achieved using decentralized estimation-and-control, where each agent uses limited communication with neighbors in a network to estimate global properties of the performance of the group. This global information is used in each agent's local motion controller, creating a feedback connection between the estimators and controllers. To ensure the stability of the coupled system, we derive a small-gain condition that can be roughly expressed as a bound on motion control gains as a function of the estimator gains and a lower bound on the algebraic connectivity of the communication network. The condition is derived for systems of agents with first-order dynamics implementing gradient controllers based on averages of the group sensor inputs. The condition is applied to two very different tasks: formation control and cooperative target localization.**

I. Introduction

Centralized control of multi-agent systems, where a single agent has access to each agent's sensory information and can issue motion commands to each agent, offers the possibility of optimal motion coordination for mobile sensor networks and swarming applications. Because this approach is not robust to faults in the central agent and does not scale well with increasing numbers of agents, most recent research in multi-agent systems has focused on decentralized reactive control. This paradigm, inspired in part by biological examples of coordinated control such as schooling fish and flocking birds [5], [12], [14], [15], has each agent make control decisions based on its own state and information it senses from the environment and nearby agents. Often the control law is based on simple gradient descent; each agent moves locally to minimize a cost J which encodes the desired group behavior. If the gradient of the objective function is *spatially distributed* over the sensing network, meaning that the local gradient can be calculated using only sensed information on neighbors, then simple gradient following can effectively yield the desired behavior [3], [4]. The set of group behaviors that can be achieved by local controllers using only immediate information about neighbors is limited, however.

We are developing a framework where each agent has access to global information on the performance of the group, allowing it to make more informed local control decisions [8], [18]. Each agent may still implement a local gradient controller, but we can now use cost functions J with local gradients that depend on *global* information. This vastly increases the set of desired group behaviors that can be encoded in J. The necessary global information is estimated by local estimators using limited communication from neighbors in a (usually sparse) communication network that may be changing with time. The amount of information communicated by each agent is independent of the number of agents. This retains the full scalability and robustness of decentralized reactive control, while gaining one of the advantages (global information) of centralized control.

The multi-agent system, then, consists of a time-varying communication network of agents each running an estimator and motion controller in parallel. This raises the question of how to ensure the stability of a network of coupled estimators and controllers. There is no simple separation principle that guarantees the stability of the entire system when the coupled estimators and coupled plant dynamics and controllers (using correct estimates) are individually stable. Instead, we need to find a time-scale separation (estimator dynamics much faster than motion dynamics) or small-gain condition to ensure the stability of the entire system.

In this paper we derive a small-gain condition for semi-global coupled stability for systems satisfying the following conditions:

- agents have single-integrator motion dynamics;
- agents implement identical gradient control laws; and
- evaluating the gradient control law requires averages of measurements over all the mobile sensors, obtained by *dynamic average consensus estimators* [7], [8], [16], which are time-varying extensions of the static average consensus estimator in [11].

Example tasks that fit this framework, and that are not solvable using decentralized reactive control, include controlling a swarm formation to achieve a set of geometric moments [8] and controlling mobile sensors to maximize their fused sensory information on the location of a moving target. Thus the formulation applies to both *action* tasks like the formation control task and *mobile sensing* tasks like the cooperative target localization task.

The small-gain condition roughly takes the form of a bound on motion control gains as a function of the estimator gains and a lower bound on the algebraic connectivity of the communication network. We believe such bounds are fundamental to a theory of decentralized coordinated control of multi-agent systems.

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We describe our general design approach in Section II and the small-gain stability condition is derived in Section III. In Section IV, we apply it to the formation control and cooperative target localization examples. Future work is briefly described in Section V.

II. DISTRIBUTED DESIGN

A. Cost Function

We have n kinematic agents, and the configuration of agent i is written $p_i = [p_i^1 \dots p_i^q]^T \in \mathbb{R}^q$. The total system configuration is $p \in \mathbb{R}^{nq}$. The cost J encoding the desired group behavior is a function of global properties f of the system. Specifically, we consider cost functions that can be written

$$J(p) = J(f(p), \beta), \tag{1}$$

where

$$f(p) = \frac{1}{n}\sum_{i=1}^{n} g(p_i) = [f^1 \cdots f^m]^T \in \mathbb{R}^m \tag{2}$$

are m pieces of global information. These are simply the average over each agent's local information $g(p_i)$, obtained from internal state and sensor measurements. Examples of f include the center of mass of a swarm in formation control [8], average sensor readings (temperature, gas concentration, etc.) in sensor networks [1] and global estimate uncertainty in target localization [2]. The constant vector β is known to each agent, and may represent common goals or controller gains. We require J to be *nonnegative* and *proper* in p (i.e., radially unbounded) so that it is a global storage function.

B. Centralized Design

In the centralized design, each agent implements a gradient controller

$$\dot{p}_i = u_i = -\frac{\partial J}{\partial p_i}. \tag{3}$$

Note that the control gains are present in the definition of the cost J; a cost αJ with $\alpha > 1$ would give agent motions in the same direction, but at a higher speed. Notice that

$$\frac{\partial J}{\partial p_i} = \begin{bmatrix} \frac{\partial f^1}{\partial p_i^1} & \cdots & \frac{\partial f^m}{\partial p_i^1} \\ \vdots & & \vdots \\ \frac{\partial f^1}{\partial p_i^q} & \cdots & \frac{\partial f^m}{\partial p_i^q} \end{bmatrix} \begin{bmatrix} \frac{\partial J}{\partial f^1} \\ \vdots \\ \frac{\partial J}{\partial f^m} \end{bmatrix} = D_i^T \begin{bmatrix} \frac{\partial J}{\partial f^1} \\ \vdots \\ \frac{\partial J}{\partial f^m} \end{bmatrix}$$

with $D_i = \frac{\partial f}{\partial p_i} \in \mathbb{R}^{m \times q}$. In general this is a centralized design; although D_i can be obtained locally for each agent i, the value of $\frac{\partial J}{\partial f}$ depends on f. This is in contrast to *spatially distributed* objective function gradients where $\frac{\partial J}{\partial p_i}$ depends only on p_i and its sensed neighbors.

This design guarantees J to be nonincreasing along trajectories in forward time. When J is nonnegative and proper, we know every trajectory is bounded. Based on LaSalle's theorem we further conclude that every trajectory converges to an equilibrium set at which $\frac{\partial J}{\partial p} = 0$. In general this set may contain local minima which are not globally minimum.

C. Consensus Estimators

Each agent can dynamically estimate the average information in (2) using a *Proportional Average Consensus Estimator* [16], [7] with local input $\phi_i = g(p_i)$. Given inputs, internal states and outputs ϕ_i, w_i, $w_i \in \mathbb{R}^m$, the P estimator is given by the following equations (see [7] for details):

$$\begin{aligned} \dot{w}_i &= -\gamma w_i - K_p \sum_{j \in \mathcal{N}_i} \left[x_i - x_j\right] && (4) \\ x_i &= w_i + \phi_i. && (5) \end{aligned}$$

Here $\gamma \geq 0$ is a forgetting factor, $\mathcal{N}_i$ contains all one-hop neighbors of agent i in the communication network, and $K_p > 0$ is the estimator gain. When the network is connected over time, each estimator output y_i will track the global signal $\frac{1}{n}\sum_{i=1}^{n} \phi_i$ with zero steady-state error for $\gamma = 0$ and no agents entering or leaving the system, or small steady-state error if the number of agents is changing and $\gamma > 0$. [7]. Since each agent only exchanges information with its neighbors, the estimation process is scalable.

D. Distributed Design

We obtain a decentralized version of the gradient control law in (3) as follows. Each agent estimates the global information f by running a P average consensus estimator with local input $\phi_i = g(p_i)$, then replaces the f in the controller with its estimator output $x_i = [x_i^1 \cdots x_i^m]^T$. After that we scale the control effort by $[I + D_i^T \Lambda_i D_i]^{-1}$, yielding

$$\dot{p}_i = u_i = -[I + D_i^T \Lambda_i D_i]^{-1} \left.\frac{\partial J}{\partial p_i}\right|_{f = x_i}. \tag{6}$$

where $D_i = \frac{\partial f}{\partial p_i} \in \mathbb{R}^{m \times q}$, $I \in \mathbb{R}^{m \times m}$ is the identity matrix and $\Lambda_i > 0$.

The foremost concern of this new distributed design is stability issues: In a stable centralized controller, when we replace the global information f with an estimator output x_i, the introduced estimation error $e_i = f - x_i$ may drive the system unstable. The positive-definite matrix $D_i^T \Lambda_i D_i$ in (6) dynamically changes the control gains to guarantee the overall stability of the feedback-connected estimation and control process. For this reason, we refer to Λ_i as a nonlinear damping matrix [10]. Larger damping, i.e., smaller control gains, improve the stability of the system, possibly at the expense of performance. Our small-gain condition specifies a lower bound on Λ_i that guarantees semi-global stability of the coupled system. The full stability analysis is presented in the next section.

III. STABILITY ANALYSIS

We define the estimator input and output variables Z, E:

$$\begin{aligned} Z &= [z_1 \cdots z_n] = [\frac{d}{dt} g(p_1) \cdots \frac{d}{dt} g(p_n)] \\ E &= [e_1 \cdots e_n] = [f \cdots f] - [x_1 \cdots x_n] \end{aligned}$$

Let $U = \mathrm{tr}(EE^T)$ be the measure of the estimation error, the behavior of the estimator is characterized in [8]:

$$\dot{U} \leqslant -\varepsilon U + \frac{1}{\varepsilon}\mathrm{tr}(ZZ^T) = \sum_{i=1}^{n}\Big[-\varepsilon |e_i|^2 + \frac{1}{\varepsilon}|z_i|^2\Big] \tag{7}$$

where $\varepsilon(t) = K_p\lambda_2(t)$ and $\lambda_2(t)$ is the algebraic connectivity [9] of the underlying communication graph Laplacian. Here we use the undirected graph given by the neighbor relations $\mathcal{N}_i$ with unit weights. For a connected network with n nodes, λ_2 reaches its minimum $\lambda_{\min} = 2 - 2\cos(\pi/n)$ [6] when the communication topology is a line graph.

Now we characterize the input-output relationship of the controller. First we find out how much the control effort changes when we replace the global information f with x_i in the controller. We can use the Mean Value theorem:

$$\left.\frac{\partial J}{\partial f^j}\right|_{f=f} - \left.\frac{\partial J}{\partial f^j}\right|_{f=x_i} = \left.\frac{\partial^2 J}{\partial f^j \partial f}\right|_{f=\widetilde{f}_{ij}} e_i \tag{8}$$

where each

$$\widetilde{f}_{ij} = f - \alpha_j(f - x_i) = f - \alpha_j e_i \qquad 0 \le \alpha_j \le 1 \tag{9}$$

is on the line segment between f and x_i. Denote the row vector $\left.\frac{\partial^2 J}{\partial f^j \partial f}\right|_{f=\widetilde{f}_{ij}}$ as C_{ij} and we call the assembled matrix $C_i = [C_{i1}^T \ \cdots \ C_{im}^T]^T$ the Hessian matrix of the system. With this notation, the change in control effort is written

$$\begin{aligned} & \left.\frac{\partial J}{\partial p_i}\right|_{f=f} - \left.\frac{\partial J}{\partial p_i}\right|_{f=x_i} \\ = & \left(\frac{\partial f}{\partial p_i}\right)^T \left(\left.\frac{\partial J}{\partial f}\right|_{f=f} - \left.\frac{\partial J}{\partial f}\right|_{f=x_i} \right) \\ = & D_i^T C_i e_i. \end{aligned} \tag{10}$$

which is the error between the centralized design and the distributed design. Then the input-output relationship of the controller is given as

$$\begin{aligned} \dot{J} &= \sum_{i=1}^n \dot{p_i}^T \frac{\partial J}{\partial p_i} \\ &= \sum_{i=1}^n \dot{p_i}^T [I + D_i^T \Lambda_i D_i][I + D_i^T \Lambda_i D_i]^{-1} (\left.\frac{\partial J}{\partial p_i}\right|_{f=x_i} \\ &\quad + D_i^T C_i e_i) \\ &= -\sum_{i=1}^n \dot{p_i}^T [I + D_i^T \Lambda_i D_i] \dot{p_i} + \sum_{i=1}^n \dot{p_i}^T D_i^T C_i e_i \\ &\le -\sum_{i=1}^n \dot{p_i}^T (I + D_i^T (\Lambda_i - \frac{C_i C_i^T}{\varepsilon}) D_i) \dot{p_i} + \sum_{i=1}^n \varepsilon |e_i|^2 \\ &= -\sum_{i=1}^n \dot{p_i}^T \dot{p_i} - \sum_{i=1}^n z_i^T (\Lambda_i - \frac{C_i C_i^T}{\varepsilon}) z_i + \sum_{i=1}^n \varepsilon |e_i|^2. \end{aligned}$$

A. The Small-Gain Condition

To combine J and U together, we define a storage function

$$\Upsilon(p, E) = J + (1+\mu)U \tag{11}$$

with $\mu > 0$. The dissipation inequality looks like:

$$\dot{\Upsilon} \leqslant -\sum_{i=1}^n \dot{p_i}^T \dot{p_i} - \sum_{i=1}^n z_i^T (\Lambda_i - \frac{C_i C_i^T + (1+\mu)I}{\varepsilon}) z_i - \sum_{i=1}^n \mu\varepsilon |e_i|^2. \tag{12}$$

Below we present main stability result of this paper.

Theorem 1: Given task $J = J(f, \beta)$ and we assume its centralized gradient design is stable and the possibly time-varying communication network remains connected. If either one of the following conditions holds:

1) the cost function J is smooth and proper in f, or
2) the global information f is globally bounded,

Then there exists a corresponding distributed design: Each agent constructs a P estimator to estimate the required global information and closes the feedback loop by adding nonlinear dampings with gains

$$\Lambda_i \ge \frac{Q + (1+\mu)I}{\lambda_{\min} K_p}. \tag{13}$$

(Q is an upperbound of $C_i C_i^T$ defined in below.) The resulting system is stable and each trajectory converges to the same equilibrium set as in the central design.

Proof: We use Υ_0 to denote the initial value of the storage function Υ. Based on its definition in (11), we know

$$\|e_i\| \le \sqrt{\frac{\Upsilon_0}{1+\mu}}, \quad J(f) \le \Upsilon_0. \tag{14}$$

In either J is proper in f or f is globally bounded, we have

$$\|e_i\| \le \sqrt{\frac{\Upsilon_0}{1+\mu}}, \quad \|f\| \le a \tag{15}$$

for some scalar a. From (9) the triangle inequality gives

$$\|\widetilde{f}_{ij}\| \le \sqrt{\frac{\Upsilon_0}{1+\mu}} + a \tag{16}$$

Additionally we have

$$C_i C_i^T \le Q = \sup_{(\widetilde{f}_{i1}, \cdots, \widetilde{f}_{im}) \in \Psi^m} C_i C_i^T < \infty \tag{17}$$

with $\Psi = [-\sqrt{\frac{\Upsilon_0}{1+\mu}} - a, \sqrt{\frac{\Upsilon_0}{1+\mu}} + a]$. Q is a function of the C_i, which is a function of the control gains. Now we choose

$$\Lambda_i \ge \frac{Q + (1+\mu)I}{\lambda_{\min} K_p} \ge \frac{Q + (1+\mu)I}{\varepsilon K_p} \tag{18}$$

so that the small gain condition in (12)

$$\varepsilon \Lambda_i \ge C_i C_i^T + (1+\mu)I \tag{19}$$

is satisfied at $t = 0$. From the Lyapunov stability theorem, $\Upsilon(t) < \Upsilon_0$ and therefore (15), (16) remain satisfied. Because (9) holds over time, we know (16) still holds and the domain Ψ will not increase. Therefore by satisfying (18), the small gain condition (19) is automatically satisfied over time and $\dot{\Upsilon}(t) \le 0$ from (12). By LaSalle's theorem, every trajectory converges to an equilibrium set at which $\dot{p} = 0$ and $E = 0$. So in steady state the estimation error vanishes and the trajectory will converge to the same equilibrium set as in the centralized design. ■

When $C_i C_i^T$ is globally bounded (e.g. J is a quadratic cost function) we get a global stability result, otherwise the result is semi-global. In general, the value of Q is obtained from a

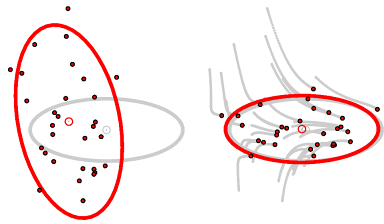

Fig. 1. (Left) The initial configuration of a swarm, a uniform-density ellipse with the same mass and first- and second-order moments as the swarm, and the goal formation of the swarm represented as another uniform-density ellipse. (Right) The swarm converges to a configuration having the desired first- and second-order moment statistics.

standard nonlinear optimization solver. It is also possible to bound C_i by taking advantage of special problem structure. We use the second approach in our case studies.

IV. Case Studies

A. Formation Control

The problem of controlling the formation shape a group of n robots is considered in [8], where moment statistics are used to characterize the shape of the group. In the case of 2-d robots (Fig 1), denote the robot positions as $p_i = (p_i^x, p_i^y)$. The first-order moments

$$f^1 = \frac{1}{n}\sum_{i=1}^{n} p_i^x, \quad f^2 = \frac{1}{n}\sum_{i=1}^{n} p_i^y$$

give the center of mass of the formation, and the second-order moments

$$f^3 = \frac{1}{n}\sum_{i=1}^{n}(p_i^x)^2, \quad f^4 = \frac{1}{n}\sum_{i=1}^{n} p_i^x p_i^y, \quad f^5 = \frac{1}{n}\sum_{i=1}^{n}(p_i^y)^2$$

give the moment of inertia of the group. Here the global information is $f = [f^1 \cdots f^5]^T$ and the goal is to move the robots to satisfied certain desired formation statistics $f^* \in \mathbb{R}^5$. A centralized design uses

$$J = (f^* - f)^T \Gamma (f^* - f) \tag{20}$$

as the cost function and the gradient-based controller is:

$$\dot{p_i} = -D_i^T \Gamma (f^* - f) \tag{21}$$

In problem formulations quadratic costs are especially favored by this design approach. First the function J as in (20) is radially unbounded and therefore proper. Additionally, $C_i = \Gamma$ is a constant matrix and the nonlinear optimization procedure is saved. Q can be chosen as $Q = \Gamma^2$. Applying Theorem 1 we choose the following control law:

$$\dot{p_i} = -[I + \frac{D_i^T K_p^{-1}(\Gamma^2 + (1+\mu)I)D_i}{\lambda_{\text{min}}}]^{-1} D_i^T \Gamma (f^* - x_i) \tag{22}$$

Theorem 1 guarantees that this coupled estimate and control system will be stable.

For the centralized design, appropriate Γ is chosen in [8] such that the equilibrium set contains only global minimum points. This implies our distributed design will also converge to the global minimum points.

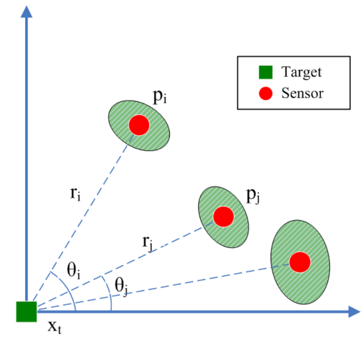

Fig. 2. Schematic of the sensor model.

B. Active Localization

The problem of active sensing using n 2-d mobile sensor nodes to estimate the state of a dynamic target is considered in [2]. The sensors and target positions are written: $p_1, \cdots, p_n, x_t \in \mathbb{R}^2$, The i^{th} sensor takes a measurement: $z_i = x_t + v_i, i = 1, \ldots, n$ and $v_i \sim N(0, \Sigma_i)$ is the measurement noise with standard Gaussian distribution. The standard range-finding sensor model [13] was used, so the covariance matrix Σ_i assumes a diagonal structure in the sensor's local range/bearing frame:

$$R_i = \begin{bmatrix} (\sigma^i_{\text{range}})^2 & 0 \\ 0 & (\sigma^i_{\text{bearing}})^2 \end{bmatrix}. \tag{23}$$

The range measurement noise variance $(\sigma^i_{\text{range}})^2$ is commonly represented by a function $h_r(r_i)$ of the distance r_i from the target to sensor i. The bearing noise variance $(\sigma^i_{\text{bearing}})^2$ also depends on the range and can be modeled as $h_b(r_i)$. We use the following form:

$$(\sigma^i_{\text{range}})^2 = h_r(r_i) = a_2(r_i - a_1)^2 + a_0 \tag{24}$$

$$(\sigma^i_{\text{bearing}})^2 = h_b(r_i) = \alpha h_r(r_i)\,, \tag{25}$$

where a_0, a_1, a_2, α are model parameters. This measurement uncertainty model assumes the existence of a "sweet spot" location $r_i = a_1$ at which the noise is at its minimum value. In practice, when the target is out of the sensing range, we can initialize the diagonal entries of R_i to be ∞.

We use $\theta_i = \angle(p_i - x_t)$ as the angle from the target to the agent i, and the rotation matrix

$$T_i = \begin{bmatrix} \cos(\theta_i) & \sin(\theta_i) \\ -\sin(\theta_i) & \cos(\theta_i) \end{bmatrix} \tag{26}$$

transforms R_i, the uncertainty matrix in the local frame, to the uncertainty matrix in the global Cartesian frame:

$$\Sigma_i = T_i R_i T_i^T. \tag{27}$$

In order to fuse the local target position measurements z_i and error covariances R_i to obtain a global target position estimate $\hat{x}_{\text{global}}$ and global error covariance P_{global}, the fol-

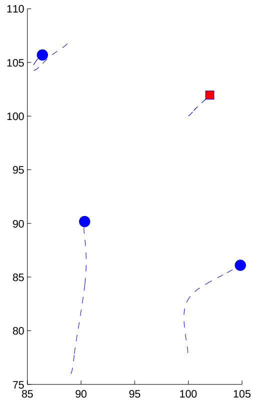

Fig. 3. Sensors move to their sweet spots and scatter around the target to get better estimates.

lowing relationships were used [2], [13]:

$$P_{\text{global}}^{-1}\hat{x}_{\text{global}} = \sum_{i=1}^{n}(T_iR_iT_i^T)^{-1}z_i = \sum_{i=1}^{n}T_iR_i^{-1}T_i^Tz_i \quad (28)$$

$$P_{\text{global}}^{-1} = \sum_{i=1}^{n}(T_iR_iT_i^T)^{-1} = \sum_{i=1}^{n}T_iR_i^{-1}T_i^T\,, \quad (29)$$

P_{global} gives a global characterization of the uncertainty of the estimate, and each sensor moves to reduce the fused uncertainty. The cost function $J = \det(P_{\text{global}})$ is used and this leads to the following gradient controller (in polar coordinates):

$$\begin{aligned} u_i^r &= -\frac{\partial J}{\partial r_i} \\ &= J\cdot\text{tr}(2a_2(r_i-a_1)T_iR_i^{-2}\begin{bmatrix}1 & 0\\ 0 & \alpha\end{bmatrix}T_i^TP_{\text{global}}) \\ u_i^\theta &= -\frac{1}{r_i}\frac{\partial J}{\partial \theta_i} = J\cdot\text{tr}\big((A_i+A_i^T)P_{\text{global}}\big)/r_i \end{aligned}$$

with

$$A_i = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}T_iR_i^{-1}T_i^T\,. \quad (30)$$

As shown in [2], this centralized controller drives the sensors to their sweet spots and space themselves from each other by 60 degrees relative to the target.

1) Choosing the global information f: Plugging (23) and (26) into (29), we get

$$\begin{aligned} \frac{1}{n}P_{\text{global}}^{-1} &= \frac{1}{n}\sum_{i=1}^{n}T_iR_i^{-1}T_i^T \\ &= \frac{1}{n}\sum_{i=1}^{n}\begin{bmatrix}\frac{\cos^2(\theta_i)+\frac{1}{\alpha}\sin^2(\theta_i)}{f_r(r_i)} & -\frac{\sin(2\theta_i)}{2f_r(r_i)}(1-\frac{1}{\alpha})\\ -\frac{\sin(2\theta_i)}{2f_r(r_i)}(1-\frac{1}{\alpha}) & \frac{\sin^2(\theta_i)+\frac{1}{\alpha}\cos^2(\theta_i)}{f_r(r_i)}\end{bmatrix}. \end{aligned}$$

Now we define the global information $f = [f^1\ f^2\ f^3]^T$ with

$$f^1 = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{f_r(r_i)} \quad (31)$$

$$f^2 = \frac{1}{n}\sum_{i=1}^{n}\frac{\cos(2\theta_i)}{f_r(r_i)} \quad (32)$$

$$f^3 = \frac{1}{n}\sum_{i=1}^{n}\frac{\sin(2\theta_i)}{f_r(r_i)} \quad (33)$$

Using triangular equalities, we obtain:

$$\frac{1}{n}P_{\text{global}}^{-1} = \begin{bmatrix}\frac{\alpha+1}{2\alpha}f^1+\frac{\alpha-1}{2\alpha}f^2 & \frac{1-\alpha}{2\alpha}f^3\\ \frac{1-\alpha}{2\alpha}f^3 & \frac{\alpha+1}{2\alpha}f^1-\frac{\alpha-1}{2\alpha}f^2\end{bmatrix} \quad (34)$$

Then the cost function can be written as

$$J = \det(P_{\text{global}}) = \frac{1}{\frac{n(\alpha-1)^2}{4\alpha^2}[(\frac{\alpha+1}{\alpha-1}f^1)^2-(f^2)^2-(f^3)^2]}. \quad (35)$$

2) Bounding the global information f: From the sensor model given in (24), it is easy to see $f_r(r_i) > a_0$. Therefore $\|f\|_2 \le \sqrt{3}|f^1| \le \frac{\sqrt{3}}{a_0}$. So we can apply Theorem 1 to find the nonlinear damping gain Λ_i and next we give a conservative lower bound on Λ_i.

3) A lower bound on the nonlinear damping gain: We start by giving an alternative strategy to bound the matrix product $C_iC_i^T$ without solving any optimization problem. Given a square matrix $C_i \in \mathbb{R}^{m\times m}$, let λ_i^* be its eigenvalue with the largest absolute value. It is easy to verify that $C_iC_i^T < (\lambda_i^*)^2I$. Furthermore, we have

$$\begin{aligned} |\lambda_i^*| &< \|C_i\|_\infty = \max_j |\left.\frac{\partial^2 J}{\partial f_j\partial f}\right|_{f=\tilde{f}_{ij}}|_\infty \\ &\le \sup_{f\in\Psi}\max_j\left|\left.\frac{\partial^2 J}{\partial f_j\partial f}\right|_{f=f}\right|_\infty = \sup_{f\in\Psi}\left\|\frac{\partial^2 J}{\partial f^2}\right\|_\infty \end{aligned} \quad (36)$$

From that we obtain the following theorem:

Proposition 1: Using local information, each agent i can calculate a lower bound of the Hessian matrix:

$$\left\|\frac{\partial^2 J}{\partial f^2}\right\|_\infty \le \frac{n(\alpha^2-1)f_r^2(r_i)[n(3\alpha^2-4\alpha+1)f_r(r_i)+\alpha a_0^2]}{2\alpha a_0^2} \quad (37)$$

Proof: See appendix. ■

Applying Theorem 1, we use the following control law:

$$\begin{bmatrix}\tilde{u}_i^r\\ \tilde{u}_i^\theta\end{bmatrix} = [I+\frac{(1+\mu+\eta_i^2)D_i^TK_p^{-1}D_i}{\lambda_{\min}}]^{-1}\begin{bmatrix}u_i^r\\ u_i^\theta\end{bmatrix} \quad (38)$$

with

$$\begin{aligned} \eta_i &= \frac{n(\alpha^2-1)f_r^2(r_i)[n(3\alpha^2-4\alpha+1)f_r(r_i)+\alpha a_0^2]}{2\alpha a_0^2} \\ u_i^r &= \det(x_i)\cdot\text{tr}(2a_2(r_i-a_1)T_iR_i^{-2}\begin{bmatrix}1 & 0\\ 0 & \alpha\end{bmatrix}T_i^Tx_i) \\ u_i^\theta &= \det(x_i)\cdot\text{tr}\big((A_i+A_i^T)x_i\big)/r_i \end{aligned}$$

Theorem 1 guarantees the stability of the coupled system.

V. Summary and Future Work

We have presented a small-gain condition ensuring the stability of cooperative multi-agent control based on decentralized estimate-and-control. This condition provides semi-global stability guarantees provided the motion control gains are sufficiently small relative to estimator gains and the algebraic connectivity of the communication network. The condition applies to systems of kinematic agents following gradients of J, where the gradient of J requires averaged information over all the agents. Example applications include formation control and cooperative target localization.

Faster convergence of the estimators permits more aggressive control gains without inducing instability. This can be achieved by increasing the estimator gain K_p. If there is communication delay, however, large estimator gains may result in estimator instability. Instead of using a single estimator gain K_p, we can assign each communication link its own gain, based on the communication network topology, to maximize the convergence rate while ensuring stability [17], [19]. Future work will explicitly incorporate bounded communication delay in the formulation of the small-gain condition.

The results in this paper assume a proportional (P) dynamic average consensus estimator. The improved PI estimator [7], [8] provides zero steady-state estimator error even when agents enter or leave the network. Future work will derive small-gain conditions using a PI estimator.

Finally, we would like to find more general small-gain conditions that apply to vehicles with second-order or under-actuated dynamics and different types of objective functions.

Appendix

Here we derive an upper bound of $\left\|\frac{\partial^2 J}{\partial f^2}\right\|_\infty$.

From (35), straightforward calculation gives :

$$\frac{\partial^2 J}{\partial f^2} = 8\rho^2 J^3 vv^T - 2\rho J^2 K \tag{39}$$

with

$$\rho = \frac{n(\alpha-1)^2}{4\alpha^2} \tag{40}$$

$$v = \begin{bmatrix} \frac{\alpha+1}{\alpha-1} f_1 & -f_2 & -f_3 \end{bmatrix}^T \tag{41}$$

$$K = \begin{bmatrix} \frac{\alpha+1}{\alpha-1} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{42}$$

Therefore,

$$\left\|\frac{\partial^2 J}{\partial f^2}\right\|_\infty \le 8\rho^2 J^3 \|vv^T\|_\infty + 2\rho J^2 \|K\|_\infty \tag{43}$$

(42) gives

$$\|K\|_\infty = \frac{\alpha+1}{\alpha-1}. \tag{44}$$

Using the fact $|f_i| \le \frac{1}{a_0}$, we get

$$\|vv^T\|_\infty \le \frac{(\alpha+1)(3\alpha-1)}{(\alpha-1)^2 a_0^2}. \tag{45}$$

To bound the global uncertainty, each agent can relate J to its own measurement uncertainty. Based on (28) we have

$$P_{\text{global}}^{-1} > T_i R_i^{-1} T_i^T > \frac{1}{\alpha f_r(r_i)} I. \tag{46}$$

Therefore

$$J = \det(P_{\text{global}}) < \alpha^2 f_r^2(r_i). \tag{47}$$

Plugging (44), (45) and (47) into (48) gives

$$\left\|\frac{\partial^2 J}{\partial f^2}\right\|_\infty \le \frac{n(\alpha^2-1) f_r^2(r_i)[n(3\alpha^2-4\alpha+1) f_r(r_i) + \alpha a_0^2]}{2\alpha a_0^2}. \tag{48}$$

For each agent i, this bound only requires local information to calculate.

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