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Controllability of Single Input Rolling Manipulation

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Abstract

This paper investigates controllability of underactuated rolling systems consisting of a smooth object rolling on a moving smooth surface. Our system consists of a spherical ball which rolls on the inside of an ellipsoidal bowl. The bowl has a single translational degree of freedom not aligned with any of its principal axes. The single control input is the bowl's acceleration in this direction. The object and contact motions are governed by a nonlinear system of equations derived from the kinematics and dynamics of rolling. Using existing results on small time local accessibility (STLA) and weakly positive Poisson stable (WPPS) vector fields, and assuming that the ball stays in the bowl, we show that the ball is globally controllable on its five-dimensional space of configurations relative to the bowl. We are currently working on motion planning algorithms with our experimental setup to control the equilibrium configuration of the ball.

1 Introduction

We are interested in single input robot systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u, \tag{1}$$

where $\mathbf{x}$ is the state, $\mathbf{f}$ is the drift vector field, u is a scalar input, and $\mathbf{g}$ is the vector field associated with the scalar input. Systems with a single control are the simplest type of useful minimalist robot system. The nonlinear coupling of the single control vector field with the drift field results in desirable controllability properties despite the limited control authority. While such systems are generally not locally controllable, they are often globally controllable. We are interested in the conditions for global controllability of general robotic systems of this form.

To begin to understand controllability and motion planning for systems of this type, we have demonstrated the global controllability of a planar body with a single bidirectional thruster (Lynch [10]). In this paper, we consider rolling manipulation with a single input. This is an example of *underactuated manipulation*. Our system consists of a spherical ball which rolls on the inside of an ellipsoidal bowl (Figure 1). The single control input is the translational acceleration of the ellipsoidal bowl along a direction which is not aligned with any of the principal axes of the ellipsoid. We show that the configuration of the ball relative to the bowl is globally controllable. In addition we use results due to Lewis [5] to show that the single input rolling system is not small time locally controllable (STLC).

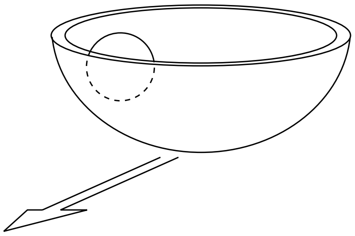

Figure 1: The rolling system.

1.1 Previous Work

Lewis and Murray [6] have investigated small time local configuration controllability (STLCC) of a class of mechanical systems whose Lagrangian is kinetic energy minus potential energy. Lewis [5] has extended these results for the case of single input mechanical control systems. That work focuses on local considerations; global controllability properties are not addressed. Goncalves [3] and Sussmann [17] have given some results for controllability of general scalar input systems which are not necessarily mechanical systems. San Martin and Crouch [15] have studied controllability of systems with compact group structure. Manikonda and Krishnaprasad [13] have given detailed results on controllability for Lie Poisson reduced dynamics. Global controllability results exist for specific systems, such as a spacecraft with momentum wheels or gas jet actuators (Crouch [2]) and a planar rigid body with a single thruster (Lynch [10]).

As far as the work on rolling systems is concerned, the kinematics of rolling has been studied extensively by Montana [16]. Some work has also been done related to controllability of kinematic rolling systems ([7], [14]), but here the systems have been considered as kinematic ones. Efficient path planning methods have been suggested for these driftless systems as it is possible to write the input vector fields in a triangular form with suitable transformations [14]. But due to the absence of a drift vector field, the above systems can never be controlled with a single input. Lynch *et al.* [12] have formulated the rolling dynamic equations and have used them for their 2-D butterfly example. Recently Jia and Erdmann [4] have studied the observability of smooth rolling objects on a plane.

2 Rolling Kinematics and Dynamics

Here we formulate the kinematics and dynamics of a rolling system. For deriving the equations we will follow these notations:

- Matrices are represented by a bold upper case letter and vectors are represented by a bold lowercase letter.
- ${}^A_B\mathbf{R}$ describes the coordinate frame B relative to the coordinate frame A.
- We refer to one of the objects as the object and the other one as the hand.
- The subscript o will denote any object variable and the subscript h will denote any hand variable.
- The subscript u and v will denote the partial derivative of the vector with respect to u and v respectively.

While formulating the system equations we assume that the hand translates without rotation. For developing the system kinematics we follow the work of Montana [16]. The configuration space of the object is defined by the product group $\Re^2 \times SO(3)$, where $\Re^2$ fixes the contact point of the object with respect to the hand and $SO(3)$ gives the orientation of the object.

We define three frames of reference relative to the object and the hand (Figure 2). A coordinate frame Π_O is attached at the center of mass of the object and is defined by the object's principal axes $\mathbf{X}_O$, $\mathbf{Y}_O$ and $\mathbf{Z}_O$. The inertia matrix $\mathbf{I}_o$ with respect to Π_O is thus diagonal. Similarly we define a frame Π_H attached to the hand with principal axes $\mathbf{X}_H$, $\mathbf{Y}_H$ and $\mathbf{Z}_H$. We also define a fixed frame Π which initially coincides with Π_O. So at the initial instant of time, the velocities expressed in the frames Π_O and Π will be identical though their derivatives will be quite different. Π_W is a base inertial frame such that the axis $\mathbf{X}_W$ is rotated with respect to $\mathbf{X}_H$ by an angle θ.

We have $S_o \subset \Re^3$ and $S_h \subset \Re^3$ as the embeddings of the surfaces of the object and the hand relative to Π_O and Π_H. There exists an open subset U of $\Re^2$ and invertible maps $\beta_{\mathbf{o}}, \beta_h : U \rightarrow S_o, S_h \subset \Re^3$. Here we consider the surface patches to be principal. This assumption is based on the fact that every point on a surface has a neighborhood that can be reparametrized as a principal patch [4]. In general $\mathbf{u}$ represents the parametric space and β is a smooth map from parametric space to Cartesian space. Let (β_o, U_o) be a right handed orthogonal coordinate system for S_o and similarly (β_h, U_h) be a right handed coordinate system for S_h. Since the patches are principal, the normalized Gauss frame at the point of contact is defined based on the following Gauss map: $\mathbf{x} = \beta_u/||\beta_u||$, $\mathbf{y} = \beta_v/||\beta_v||$ and $\mathbf{z} = \beta_u \times \beta_v/(||\beta_u|| \cdot ||\beta_v||)$ (subscript o and h used accordingly). Here $\mathbf{z}$ is chosen to be the outward normal. The coordinate frames (β_o, U_o) and (β_h, U_h) induce a normalized Gauss frame at all points on S_o and S_h. The contact point on the object O denoted by $\mathbf{q}$ is given by $\beta_o(u_o, v_o)$. The normalized Gauss frame for the object Π_o at the point of contact is well defined by the axes $\mathbf{x}_o$, $\mathbf{y}_o$ and $\mathbf{z}_o$. The orientation of the contact frame Π_o with respect to the body frame of the object Π_O is given by a 3×3 rotation matrix ${}^O_o\mathbf{R} = (\mathbf{x}_o \;\; \mathbf{y}_o \;\; \mathbf{z}_o)$. The contact point in the hand in the frame Π_H $(\mathbf{p})$ is given by $\beta_h(u_h, v_h)$. We attach to $\mathbf{p}$ a frame Π_h (with axes $\mathbf{x}_h$, $\mathbf{y}_h$ and $\mathbf{z}_h$) which is coincident with the Gaussian frame defined on S_h (direction of $\mathbf{z}_h$ is chosen to be opposite to that of $\mathbf{z}_o$). The angle of rotation needed to align $\mathbf{x}_o$ with $\mathbf{x}_h$ is given by ϕ. Accordingly Π_o is related to Π_h by a 3×3 rotation matrix:

$${}^h_o\mathbf{R} = \begin{bmatrix} \mathbf{R}_\phi & 0 \\ 0 & -1 \end{bmatrix} \text{ and } \mathbf{R}_\phi = \begin{bmatrix} cos(\phi) & -sin(\phi) \\ -sin(\phi) & -cos(\phi) \end{bmatrix}$$

Consequently the orientation of the object frame Π_O relative to the contact frame of the hand Π_h is given by:

$${}^h_O\mathbf{R} = {}^h_o\mathbf{R}\,{}^O_o\mathbf{R}^T$$

The body frame of the hand Π_H is related to the contact frame of the hand Π_h by the rotation matrix ${}^H_h\mathbf{R} = (\mathbf{x}_h \;\; \mathbf{y}_h \;\; \mathbf{z}_h)$. So the transformation required

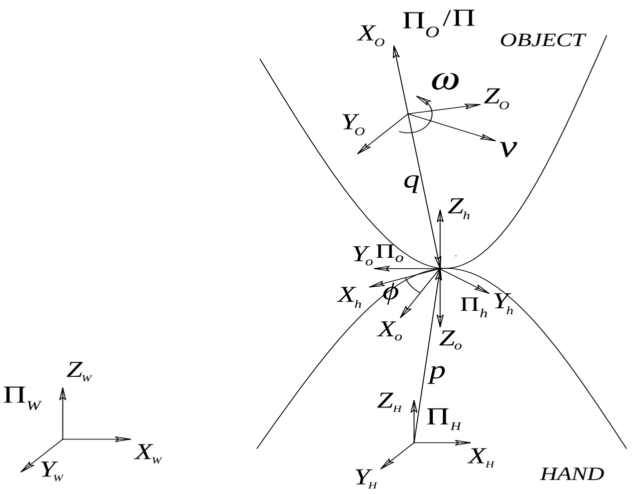

Figure 2: Rolling contact definitions.

to align the object frame Π_O with the hand frame Π_H is:

$$ {}^H_O\mathbf{R} = {}^H_h\mathbf{R}\,{}^h_o\mathbf{R}\,{}^O_o\mathbf{R}^T \qquad (2)$$

The contact kinematics depends on the relative motion between the two contact frames Π_o and Π_h. The angular velocity of Π_o relative to Π_h is denoted by the vector $(\Omega_x, \Omega_y, \Omega_z)^T$. The linear and angular velocities of the object relative to the frame Π_O is given by $\mathbf{v}$ and $\boldsymbol{\omega}$ respectively. Since Π_o is fixed relative to Π_O, we have

$$\boldsymbol{\omega} = {}^O_o\mathbf{R}^T (\Omega_x, \Omega_y, \Omega_z)^T$$

We also define the shape operator $(\mathbf{K})$ as $\mathbf{K} = [\mathbf{x}, \mathbf{y}]^T [\mathbf{z}_u/\|\beta_u\|, \mathbf{z}_v/\|\beta_v\|]$. The geodesic curvature or the torsion form is defined as $\mathbf{T} = \mathbf{y}^T[\mathbf{x}_u/\|\beta_u\|, \mathbf{x}_v/\|\beta_v\|]$, and we associate a metric $\mathbf{M}$ as the 2×2 diagonal matrix $\mathbf{M} = diag(\|\beta_u\|, \|\beta_v\|)$.

The equations describing the system kinematics for two three dimensional objects in contact has been fully developed by Montana [16]. We impose the rolling without slipping condition which implies that there will be no translational relative velocity at the contact point. We use the hard contact model where the contact can only transmit three components of forces. With these conditions the kinematic equations of [16] can be written as:

$$\dot{\mathbf{u}}_o = \mathbf{M}_o^{-1}(\mathbf{K}_o + \tilde{\mathbf{K}}_h)^{-1}(\mathbf{y}_o, -\mathbf{x}_o)^T\boldsymbol{\omega} \qquad (3)$$

$$\dot{\mathbf{u}}_h = \mathbf{M}_h^{-1}\mathbf{R}_\phi(\mathbf{K}_o + \tilde{\mathbf{K}}_h)^{-1}(\mathbf{y}_o, -\mathbf{x}_o)^T\boldsymbol{\omega} \qquad (4)$$

$$\dot{\phi} = \omega_z + \mathbf{T}_o\mathbf{M}_o\dot{\mathbf{u}}_o + \mathbf{T}_h\mathbf{M}_h\dot{\mathbf{u}}_h \qquad (5)$$

and $\tilde{\mathbf{K}}_h = \mathbf{R}_\phi\mathbf{K}_h\mathbf{R}_\phi$. Equation 5 can be modified by substituting the values of $\dot{\mathbf{u}}_o$ and $\dot{\mathbf{u}}_h$ from equations 3 and 4 respectively. Finally the simplified equation turns out to be:

$$\dot{\phi} = \mathbf{z}^T\boldsymbol{\omega} + [\mathbf{T}_o + \mathbf{T}_h\mathbf{R}_\phi](\mathbf{K}_o + \tilde{\mathbf{K}}_h)^{-1}](\mathbf{y}_o, -\mathbf{x}_o)^T\boldsymbol{\omega}. \qquad (6)$$

Now coming to the dynamics of rolling, we follow the method adopted by Jia and Erdmann [4]. The dynamics of the object is governed by Newton's and Euler's equations. To satisfy Newton's equation, we consider the force equilibrium of the object O (the force and moment equilibrium equations has been expressed in terms of the body coordinate frame).

Newton's Equation:

$$\mathbf{F} + m_o(\mathbf{z}_o)g = m(\dot{\mathbf{v}} + \boldsymbol{\omega} \times \mathbf{v}) \qquad (7)$$

where m_o is the mass of the object, $\mathbf{F}$ is the force at the point of contact, g is the gravitational acceleration and $\mathbf{z}_o$ is the direction of the $\mathbf{z}$ axis of the object Gauss frame at the point of contact. For Euler's equation, we consider the moment equilibrium of the object O about its center of mass.

Euler's Equation:

$$\beta_o \times \mathbf{F} = \mathbf{I}_o\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}_o\boldsymbol{\omega} \qquad (8)$$

where $\mathbf{I}_o$ is the inertia matrix of the object in terms of its body frame and $\beta_o \times \mathbf{F}$ is the applied torque about the center of mass of the object. The no-slip constraint gives rise to the following equation:

$${}^H_O\mathbf{R}(\mathbf{v} + \boldsymbol{\omega} \times \beta_o) = \mathbf{v}_h \qquad (9)$$

To arrive at the final equations describing the system dynamics, we eliminate $\mathbf{F}$ from equations 7 and 8, differentiate equation 9, and back substitute $\mathbf{v}$ and $\dot{\mathbf{v}}$. With these algebraic manipulations we get:

$$\begin{aligned} \dot{\boldsymbol{\omega}} \;=\; & D^{-1}\beta_o \times {}^O_h\mathbf{R}^T\mathbf{a}_h - D^{-1}(\beta_o \times (g\mathbf{z}_o + \\ & \boldsymbol{\omega} \times {}^O_h\mathbf{R}^T\mathbf{v}_h - \boldsymbol{\omega} \times \beta_o + 2\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \beta_o) + \\ & \boldsymbol{\omega} \times \dot{\beta}_o) + \boldsymbol{\omega} \times (\mathbf{I}_o/m)\boldsymbol{\omega}), \qquad (10) \end{aligned}$$

where $D = \mathbf{I}_o/m + {\beta_o}^2 I_3 - \beta_o{\beta_o}^T$ defines a nonsingular positive definite matrix, $\mathbf{v}_h$ is the velocity of the hand relative to Π_H, and the input $\mathbf{a}_h$ is the hand acceleration expressed in frame Π_H. The equations 3, 4, 6 and 10 govern the object motion as well as the contact motions on the object and in the hand. For details regarding dynamics of similar rolling systems, see the work of Jia and Erdmann [4].

From now on we refer to the ellipsoidal bowl as the hand and the spherical ball which rolls inside the ellipsoidal bowl as the object. Since the hand frame Π_H is rotated with respect to the global frame Π_W

and the single input of the system expressed in Π_W is $[a_w, 0, 0]^T$, $\mathbf{a}_h$ is related to a_w by

$$\mathbf{a}_h = {}^H_W\mathbf{R}[a_w, 0, 0]^T \tag{11}$$

The velocity of the hand in world frame ($[v_w, 0, 0]^T$) is related to $\mathbf{v}_h$ in a way similar to equation 11. We refer to the ball and ellipsoid rolling system as Σ.

3 Controllability

Before studying the controllability of the rolling system, we list a few formal definitions related to Poisson stability, the Lie algebra rank condition (LARC), and small time local accessibility (STLA).

3.1 Poisson Stability and Controllability

Consider a smooth vector field $\mathbf{X}$ on a smooth connected manifold M. Its flow is denoted by ϕ^X:

$$\phi^X : \Re \times M \to M; (t, p) \mapsto \phi^X_t(p)$$

Here we assume that $\phi^X_t(p)$ is defined for all time $t \geq 0$.

Definition 1 *A point p in M is said to be* positively Poisson stable *for X if for all $T > 0$ and any neighborhood U_p of p there exists a time $t > T$ such that $\phi^X_t(p) \in U_p$. The vector field X is positively Poisson stable if the set of positively Poisson stable points for X is dense in M.*

Definition 2 *A point $p \in M$ is called a* nonwandering point *of X if for all $T > 0$ and any neighborhood U_p of p, there exists a time $t > T$ such that $\phi^X_t(U_p) \cap U_p \neq 0$, where $\phi^X_t(U_p) = \{\phi^X_t(q) | q \in U_p\}$.*

One should observe here that though it is a sufficient condition that the nonwandering set of a positively Poisson stable vector field is the entire manifold M, there exist weaker conditions under which the nonwandering set is M. This gives rise to the definition of a weakly positively Poisson stable (WPPS) vector field [8].

Definition 3 *The vector field X is called* weakly positively Poisson stable *if its nonwandering set is M.*

Definition 4 *[1] For a volume-preserving flow for any region D, volume of $g^t D$ = volume of D where g^t is the phase flow at time t.*

Liouville's theorem states that the flow generated by a time independent Hamiltonian system is volume-preserving. From Poincare's recurrence theorem we know that for a volume-preserving, bijective continuous map g on a bounded region $D \subset \Re^n$, every neighborhood U of each point in D contains a point q which returns to U after repeated application of the mapping, i.e. $g^n(q) \in U$ for some positive integer n. Combining the above observations the following lemma is provided by Lian *et al.* [8]:

Lemma 1 *A time-independent Hamiltonian vector field on a bounded manifold is* WPPS.

For systems of the form of equation 1, we define $R^V(\mathbf{x}_0, \dot{\mathbf{x}_0}, T)$ to be the reachable set from the initial configuration and velocity $(\mathbf{x}_0, \dot{\mathbf{x}_0})$ at time $T > 0$ by feasible trajectories remaining in the neighborhood V of $(\mathbf{x}_0, \dot{\mathbf{x}_0})$ at times $t \in [0, T]$. Then system 1 is STLA from $(\mathbf{x}_0, \dot{\mathbf{x}_0})$ if $R^V(\mathbf{x}_0, \dot{\mathbf{x}_0}, \leq T)$ contains a non-empty open set of the tangent bundle for any neighborhood V of $(\mathbf{x}_0, \dot{\mathbf{x}_0})$ for all $T > 0$. If $R^V(\mathbf{x}_0, \dot{\mathbf{x}_0}, \leq T)$ contains a neighborhood of $(\mathbf{x}_0, \dot{\mathbf{x}_0})$ for any neighborhood V and all $T > 0$, then the system is STLC. System 1 is controllable from $(\mathbf{x}_0, \dot{\mathbf{x}_0})$ if for any $(\mathbf{x}_1, \dot{\mathbf{x}_1}) \in TC$, there exists a finite time T such that $(\mathbf{x}_1, \dot{\mathbf{x}_1}) \in R^{TC}(\mathbf{x}_0, \dot{\mathbf{x}_0}, \leq T)$, where TC is the state space or the tangent bundle. STLCC is a weaker property than STLC and is defined by Lewis and Murray [6].

The accessibility Lie algebra $\mathcal{L}$ is the smallest subalgebra containing $\mathbf{f}$ and $\mathbf{g}$. The tangent vectors of $\mathcal{L}$ at the point p are denoted $\mathcal{L}(p)$. The system is said to satisfy the LARC at point p if

$$span(\mathcal{L}(p)) = T_pM \qquad \forall p \in M,$$

where T_pM is the tangent space at p. For a general driftless kinematic system with bilateral inputs, one can show with the help of Chow's theorem that the satisfaction of LARC implies local controllability. Once the dynamics are included, the above condition doesn't hold true and so proving controllability is usually much harder than proving accessibility.

For proving controllability of systems which include dynamics, one can make use of the following result given by Lian *et al.* [8].

Lemma 2 *If the drift vector field $\mathbf{f}$ is WPPS, then the system 1 is controllable if and only if it satisfies the LARC condition.*

Earlier versions of this theorem required the drift vector field $\mathbf{f}$ to be positively Poisson stable [9] and this result has been used in [2]. The above two lemmas can be synthesized in the following manner:

Theorem 1 *Consider a control system on M*

$$\dot{\mathbf{x}} = X_H + \sum_{i=1}^{n} \mathbf{g}_i \mathbf{u}_i$$

where X_H is an analytic Hamiltonian vector field and $\mathbf{g}_i$, $\forall i = 1, ..., n$ (for scalar input systems, $n = 1$), are analytic vector fields on M. Assuming M to be a bounded manifold, the system is controllable iff LARC *is satisfied.*

3.2 LARC for the Rolling System

The first step in proving controllability for the rolling system is to show that the system is STLA. For a general nonlinear system STLA is proved by showing that the system satisfies the LARC. While calculating the Lie brackets, we simplify the equations describing the system kinematics and dynamics.

Kinematics:

$$\dot{\mathbf{u}}_o = \mathbf{A}_1(\mathbf{u}_o, \mathbf{u}_h, \phi) \cdot \boldsymbol{\omega} \tag{12}$$

$$\dot{\mathbf{u}}_h = \mathbf{A}_2(\mathbf{u}_o, \mathbf{u}_h, \phi) \cdot \boldsymbol{\omega} \tag{13}$$

$$\dot{\phi} = \mathbf{A}_3(\mathbf{u}_o, \mathbf{u}_h, \phi) \cdot \boldsymbol{\omega} \tag{14}$$

Dynamics:

$$\dot{\boldsymbol{\omega}} = \mathbf{b}(\mathbf{u}_o, \mathbf{u}_h, v_w, \phi, \boldsymbol{\omega}) + \mathbf{C}(\mathbf{u}_o, \mathbf{u}_h, \phi)[a_w, 0, 0]^T \tag{15}$$

where $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{b}$ and $\mathbf{C}$ are given by equations 3, 4, 6 and 10. We also add two more state equations related to the position (x_w) and velocity (v_w) of the hand (described in the world frame Π_w).

$$\dot{x}_w = v_w \quad \text{and} \quad \dot{v}_w = a_w \tag{16}$$

One should note that the input can be treated as a scalar a_w and so the effective input vector field turns out to be the first column of the matrix $\mathbf{C}$ (we will denote that by $\mathbf{c}_1$). Now, if we write equations 12, 13, 14, 15 and 16 in the form of equation 1, the drift vector field $\mathbf{f}$, the control vector field $\mathbf{g}$ and the state vector $\mathbf{x}$ takes the following form.

$$\mathbf{f} = \begin{bmatrix} \mathbf{A} \cdot \boldsymbol{\omega} \\ v_w \\ \mathbf{b} \\ 0 \end{bmatrix}, \ \mathbf{g} = \begin{bmatrix} \mathbf{0}_5 \\ 0 \\ \mathbf{c}_1 \\ 1 \end{bmatrix} \ \text{and } \mathbf{x} = \begin{bmatrix} \mathbf{u}_o \\ \mathbf{u}_h \\ \phi \\ x_w \\ \boldsymbol{\omega} \\ v_w \end{bmatrix}$$

where $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1(\mathbf{u}_o, \mathbf{u}_h, \phi), \mathbf{A}_2(\mathbf{u}_o, \mathbf{u}_h, \phi), \mathbf{A}_3(\mathbf{u}_o, \mathbf{u}_h, \phi) \end{bmatrix}^T$$

and $\mathbf{0}_m$ is a 0 vector of dimension m. Firstly, we evaluate the Lie brackets at the zero velocity section $Z_{\mathbf{x}}$ and then project it on the 8 dimensional state space ($\Re^2 \times \Re^2 \times S^1 \times \Re^3$) defining the object's configuration and velocity (omitting x_w and v_w from $\mathbf{x}$). For that purpose we consider the following Lie brackets: $\mathbf{g}$, $[\mathbf{f}, \mathbf{g}]$, $[\mathbf{g}, [\mathbf{f}, \mathbf{g}]]$, $[\mathbf{f}, [\mathbf{g}, [\mathbf{f}, \mathbf{g}]]]$, $[\mathbf{g}, [\mathbf{f}, [\mathbf{g}, [\mathbf{f}, \mathbf{g}]]]]$, $[\mathbf{f}, [\mathbf{g}, [\mathbf{f}, [\mathbf{g}, [\mathbf{f}, \mathbf{g}]]]]]$, $[\mathbf{g}, [\mathbf{f}, [\mathbf{g}, [\mathbf{f}, [\mathbf{g}, [\mathbf{f}, \mathbf{g}]]]]]]$ and $[\mathbf{f}, [\mathbf{g}, [\mathbf{f}, [\mathbf{g}, [\mathbf{f}, [\mathbf{g}, [\mathbf{f}, \mathbf{g}]]]]]]]$. After computing the Lie brackets at the zero velocity section $Z_{\mathbf{x}}$ and writing it in the form of a matrix we get the following 8×8 matrix:

$$\begin{bmatrix} \mathbf{0}_5 & -\mathbf{A} \cdot \mathbf{c}_1 & \mathbf{0}_5 & -\mathbf{A} \cdot \mathbf{a}_5 & \mathbf{0}_5 & -\mathbf{A} \cdot \mathbf{a}_7 & \mathbf{0}_5 & -\mathbf{A} \cdot \mathbf{a}_9 \\ \mathbf{c}_1 & \mathbf{0}_3 & \mathbf{a}_5 & \mathbf{0}_3 & \mathbf{a}_7 & \mathbf{0}_3 & \mathbf{a}_9 & \mathbf{0}_3 \end{bmatrix} \tag{17}$$

where $\mathbf{a}_4$, $\mathbf{a}_5$, $\mathbf{a}_6$, $\mathbf{a}_7$, $\mathbf{a}_8$, $\mathbf{a}_9$ and $\mathbf{a}_{10}$ can be written as:

$$\mathbf{a}_4 = \frac{\partial \mathbf{c}_1}{\partial \mathbf{u}_o} \cdot \mathbf{A}_1 \quad + \frac{\partial \mathbf{c}_1}{\partial \mathbf{u}_h} \cdot \mathbf{A}_2 \quad + \frac{\partial \mathbf{c}_1}{\partial \phi} \cdot \mathbf{A}_3 \quad - \frac{\partial \mathbf{b}}{\partial} \cdot \mathbf{c}_1 - \frac{\partial \mathbf{b}}{\partial v_w}$$

$$\mathbf{a}_5 = \frac{\partial \mathbf{a}_4}{\partial} \cdot \mathbf{c} + \frac{\partial \mathbf{a}_4}{\partial v_w} + \frac{\partial \mathbf{c}_1}{\partial \mathbf{u}_o} \cdot \mathbf{A}_1 \mathbf{c}_1 + \frac{\partial \mathbf{c}_1}{\partial \mathbf{u}_h} \cdot \mathbf{A}_2 \mathbf{c}_1 + \frac{\partial \mathbf{c}_1}{\partial \phi} \cdot \mathbf{A}_3 \mathbf{c}_1$$

$$\mathbf{a}_6 = \frac{\partial \mathbf{a}_5}{\partial \mathbf{u}_o} \cdot \mathbf{A}_1 \quad + \frac{\partial \mathbf{a}_5}{\partial \mathbf{u}_h} \cdot \mathbf{A}_2 \quad + \frac{\partial \mathbf{a}_5}{\partial \phi} \cdot \mathbf{A}_3 \quad - \frac{\partial \mathbf{b}}{\partial} \cdot \mathbf{a}_5$$

$$\mathbf{a}_7 = \frac{\partial \mathbf{a}_6}{\partial} \cdot \mathbf{c}_1 + \frac{\partial \mathbf{a}_6}{\partial v_w} + \frac{\partial \mathbf{c}_1}{\partial \mathbf{u}_o} \cdot \mathbf{A}_1 \mathbf{c}_1 + \frac{\partial \mathbf{c}_1}{\partial \mathbf{u}_h} \cdot \mathbf{A}_2 \mathbf{c}_1 + \frac{\partial \mathbf{c}_1}{\partial \phi} \cdot \mathbf{A}_3 \mathbf{c}_1$$

$$\mathbf{a}_8 = \frac{\partial \mathbf{a}_7}{\partial \mathbf{u}_o} \cdot \mathbf{A}_1 \quad + \frac{\partial \mathbf{a}_7}{\partial \mathbf{u}_h} \cdot \mathbf{A}_2 \quad + \frac{\partial \mathbf{a}_7}{\partial \phi} \cdot \mathbf{A}_3 \quad - \frac{\partial \mathbf{b}}{\partial} \cdot \mathbf{a}_7$$

$$\mathbf{a}_9 = \frac{\partial \mathbf{a}_8}{\partial} \cdot \mathbf{c}_1 + \frac{\partial \mathbf{a}_8}{\partial v_w} + \frac{\partial \mathbf{c}_1}{\partial \mathbf{u}_o} \cdot \mathbf{A}_1 \mathbf{c}_1 + \frac{\partial \mathbf{c}_1}{\partial \mathbf{u}_h} \cdot \mathbf{A}_2 \mathbf{c}_1 + \frac{\partial \mathbf{c}_1}{\partial \phi} \cdot \mathbf{A}_3 \mathbf{c}_1$$

From the structure of the Lie brackets it is clear that we can conclude about the LARC just by examining the rank of the simplified matrix $[\mathbf{c}, \mathbf{a}_5, \mathbf{a}_7, \mathbf{a}_9]$. To analyze the linear dependency of the individual columns of the given matrix we will get 7 systems of partial differential equations (PDE) which checks whether any of the column vector can be expressed as linear combination of all other vectors. If any of the above PDE's are satisfied then the system will violate the LARC condition. All of the above equations are PDE's of at least order 2, and we postulate that these PDE systems will generically have no solutions. Here $\det[\mathbf{c}, \mathbf{a}_5, \mathbf{a}_7, \mathbf{a}_9] = 0$ defines a lower dimensional manifold of the configuration space, so the rolling system is small-time accessible from a generic configuration.

Hypothesis 1 *The Lie brackets (17) span the full state space except for some non-generic shapes of object and hand.*

For our system Σ, we show a sample calculation in which the matrix $[\mathbf{c}, \mathbf{a}_5, \mathbf{a}_7, \mathbf{a}_9]$ has full rank at equilibrium (when the sphere rests at the bottom of the ellipsoid). Here we can represent the sphere with the

following map: $f: \mathbf{u} \rightarrow \Re^3$,
$(u, v) \rightarrow (Rcos(u)cos(v), -Rcos(u)sin(v), Rsin(u))$ where $\{(u, v) \mid -\pi/2 < u < \pi/2, -\pi < v < \pi\}$ and R is the radius of the sphere. The Gauss frame of the sphere is

$$\mathbf{x}_o = \begin{bmatrix} -sin(u)cos(v) \\ sin(u)sin(v) \\ cos(u) \end{bmatrix}, \mathbf{y}_o = \begin{bmatrix} -sin(v) \\ -cos(v) \\ 0 \end{bmatrix}$$

and

$$\mathbf{z}_o = \begin{bmatrix} cos(u)cos(v) \\ -cos(u)sin(v) \\ sin(u) \end{bmatrix}$$

The curvature form $\mathbf{K}_o$, torsion form $\mathbf{T}_o$ and the metric $\mathbf{M}_o$ are:

$$\mathbf{K}_o = \begin{bmatrix} 1/R & 0 \\ 0 & 1/R \end{bmatrix}, \mathbf{T}_o = \begin{bmatrix} 0 & -tan(u)/R \end{bmatrix}$$

$$\mathbf{M}_o = \begin{bmatrix} R & 0 \\ 0 & Rcos(u) \end{bmatrix}$$

respectively. The mapping for the ellipsoid or the hand is given by: $f: \mathbf{u} \rightarrow \Re^3$,
$(u, v) \rightarrow (acos(u)cos(v), -bcos(u)sin(v), csin(u))$ where $\{(u, v) \mid -\pi/2 < u < \pi/2, -\pi < v < \pi\}$ and a, b and c are half the length of the principal axes of the ellipsoid. The expressions for Gauss frame at the point of contact, curvature form, torsion form and the metric form can be evaluated in a similar way. For our numerical calculation we have chosen constants corresponding to our experimental setup, $a = 15cm, b = 22cm, c = 25cm$ and $R = 2.8cm$. With these values we have evaluated the Lie brackets at the equilibrium condition. We also considered the principal axis of the ellipsoid to make an angle of 30° with direction of acceleration of the ellipsoid. For the sphere ellipsoid case the matrix $[\mathbf{c}, \mathbf{a}_5, \mathbf{a}_7, \mathbf{a}_9]$ is of rank 4, which implies the system is STLA from equilibrium.

Proposition 1 *The rolling system Σ is STLA with a single input at equilibrium when the ball rests on the bottom of the ellipsoid.*

Hypothesis 2 *The rolling system Σ is STLA on the zero velocity section $Z_{\mathbf{x}}$ with a single input.*

Now after showing that the system is accessible on $Z_{\mathbf{x}}$, we calculate the Lie brackets for all non-zero velocities. After writing the Lie brackets in the form of an 8×8 matrix, we get

$$\begin{bmatrix} \mathbf{0}_5 & -\mathbf{A}\cdot\mathbf{c}_1 & \mathbf{0}_5 & -\mathbf{A}\cdot\mathbf{a}_5 & \mathbf{0}_5 & -\mathbf{A}\cdot\mathbf{a}_7 & \mathbf{0}_5 & -\mathbf{A}\cdot\mathbf{a}_9 \\ \mathbf{c} & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & \mathbf{a}_7 & \mathbf{a}_8 & \mathbf{a}_9 & \mathbf{a}_{10} \end{bmatrix}$$

We can show that the matrix will span the 8 dimensional state space of the object for non-zero velocities if the system is STLA on $Z_{\mathbf{x}}$.

Theorem 2 *The rolling system Σ is STLA for all non-zero velocities if it satisfies the LARC condition on the zero velocity section $Z_{\mathbf{x}}$.*

Proof: Consider the matrix formed by the Lie brackets at non-zero velocity. The odd numbered columns (columns 1, 3, 5 and 7) have a structure similar to that of the brackets evaluated at $Z_{\mathbf{x}}$. We can say that the odd numbered columns will be linearly independent if $\mathbf{c}_1$, $\mathbf{a}_5$, $\mathbf{a}_7$ and $\mathbf{a}_9$ are linearly independent or in other words if the system is STLA on $Z_{\mathbf{x}}$. The even numbered columns (columns 2, 4, 6 and 8) have a different structure. Based on that particular structure, we can say that the even numbered columns will be linearly independent if $\mathbf{c}$, $\mathbf{a}_5$, $\mathbf{a}_7$ and $\mathbf{a}_9$ are linearly independent.

3.3 WPPS for the Rolling System

To show that the drift field satisfies WPPS, we observe that it defines a volume-preserving flow and use Poincaré's recurrence theorem. To see that the phase flow is volume-preserving, consider that the normal force at the point of contact preserves the contact constraint and the friction force preserves rolling constraint. Neither of these constraint-preserving forces does work, so phase volume is preserved.

Lemma 3 *The flow generated by the drift vector field for the rolling system is volume-preserving.*

The next step is to show that the drift satisfies Poincare's recurrence theorem. We define a bounded region D such that the total energy of the ball in a frame moving with the bowl's current velocity is bounded by E_{max}, the potential energy of the ball at the top of the bowl. The linear and angular velocities of the ball with respect to the hand are

$$\mathbf{v}_{rel} = {}^H_O\mathbf{R}(\boldsymbol{\omega} \times \beta_o) \text{ and } \boldsymbol{\omega}_{rel} = {}^H_O\mathbf{R}\boldsymbol{\omega}. \qquad (18)$$

Now the bounded region D is defined as:

$$\begin{aligned} D = & \{(u_h, v_h, \mathbf{G}, \boldsymbol{\omega}) \mid 1/2 m_o {\mathbf{v}_{rel}}^2 + \\ & 1/2 {\boldsymbol{\omega}_{rel}}^T ({}^H_O\mathbf{R}^T) \mathbf{I}_o ({}^H_O\mathbf{R}) \boldsymbol{\omega}_{rel} \\ & + m_o g h < E_{max}\}, \qquad (19) \end{aligned}$$

where h is the height of the ball with respect to the ellipsoid and can be expressed as a function of the contact parameters of the hand (u_h, v_h) and

Figure 3: The experimental setup.

$\mathbf{G} \in SO(3)$ defines the orientation of the object. D is the bounded subset of the ball state space (relative to the bowl) such that the ball's total energy is less than E_{max}. Thus the drift field yields a volume-preserving, continuous, bijective map on a bounded region of the ball's state space (ignoring the motion of the bowl), and Poincaré's recurrence theorem is satisfied. From Hypothesis 2 and Theorem 2 the system Σ satisfies LARC at all velocities. By Lemma 2 we can conclude that our single input rolling system is controllable on D.

Theorem 3 *The single input rolling system consisting of a spherical ball rolling on the inside of an ellipsoidal bowl is globally controllable on the subset D of its* 8 *dimensional state space (using Hypothesis 2).*

3.4 STLC for the Rolling System

Lewis [5] has shown that single input simple mechanical systems are never STLCC. This implies that the system is not STLC at zero velocity.

Theorem 4 *The rolling system* Σ *is not STLC.*

Proof: If the system is STLA then $\mathbf{g}$, $[\mathbf{f},\mathbf{g}]$ $[\mathbf{g},[\mathbf{f},\mathbf{g}]]$ and some other higher order brackets are linearly independent. The bracket term $[\mathbf{g},[\mathbf{f},\mathbf{g}]]$ is the only level 2 primitive bracket between the drift and input vector fields [6]. But to satisfy the STLC requirement, in addition to being STLA, $[\mathbf{g},[\mathbf{f},\mathbf{g}]]$ has to be expressed as a combination of lower order brackets and this violates the STLA condition. So the system is not STLC at zero velocity.

4 Conclusion

In this paper we have studied the controllability properties of a single input rolling system. Our next step is motion planning and control of the equilibrium configuration of a ball in our experimental setup (Figure 3). The bowl is milled out of an aluminum block and mounted on a linear slide. Details of the system can be found at `http://lims.mech.nwu.edu/~lynch`.

This system is an example of a controllable robotic system with a single input. Other examples include a hovercraft [10] and planar juggling [11]. Future work is aimed at characterizing the set of single input robotic systems which are controllable and generating control strategies for these systems.

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