

Controllability of a Planar Body with Unilateral Thrusters

Kevin M. Lynch
Mechanical Engineering Department
Northwestern University
2145 Sheridan Road
Evanston, IL 60208-3111 USA

November 11, 1997; revised April 28, 1998

Abstract

This note investigates the minimal number of unilateral thrusters required for different versions of nonlinear controllability of a planar rigid body. For one to three unilateral thrusters, we get a new property with each additional thruster: one thruster yields small-time accessibility on the body's state space $TSE(2)$; two thrusters yield global controllability on $TSE(2)$; and three thrusters yield small-time local controllability at zero velocity states.

1 Introduction

In this note we study the minimal number of unilateral thrusters required for different versions of nonlinear controllability of a planar rigid body. The dynamics can be viewed as a simple model of a planar spacecraft or hovercraft, also studied by Manikonda and Krishnaprasad [7] and Lewis and Murray [4]. The configuration space of the body is $\mathcal{C} = SE(2)$, the set of planar positions and orientations, and its state space is the tangent bundle $T\mathcal{C}$. The configuration of the planar body is \mathbf{q} and its state is $(\mathbf{q}, \dot{\mathbf{q}})$.

We place the following restrictions on the thrusters:

1. Each thruster provides a line of force fixed in the body frame.
2. Each thruster is *unilateral*. A pair of opposing thrusters is counted as two thrusters.
3. Each thruster has only two states, off or on, with thrust magnitudes 0 or 1.
4. Only one thruster may be on at a time.

With these restrictions on the thrusters, we can choose thruster configurations verifying the following properties (which will be made formal later):

One thruster The planar body is *small-time accessible*. For any $(\mathbf{q}, \dot{\mathbf{q}})$ and any neighborhood V of $(\mathbf{q}, \dot{\mathbf{q}})$, the body can reach a full-dimensional subset of $T\mathcal{C}$ without leaving V .

Two thrusters The planar body is *controllable*. It can reach any $(\mathbf{q}_1, \dot{\mathbf{q}}_1)$ from any other $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$ in finite time.

Three thrusters The planar body is *small-time locally controllable* at zero velocity states.

These properties are tight—a planar body with one thruster can never be controllable, and a planar body with two thrusters can never be small-time locally controllable. These properties are also tight if we relax restrictions 3 and 4 on the thrusters, allowing simultaneous use of multiple thrusters with thrust values in $[0, 1]$.

2 Definitions

A coordinate frame \mathcal{F}_B is attached to the center of mass of the planar body \mathcal{B} , and its configuration in an inertial frame \mathcal{F}_W is given by $\mathbf{q} = (x_w, y_w, \theta_w)^T$. The state of \mathcal{B} is written $(\mathbf{q}, \dot{\mathbf{q}}) \in T\mathcal{C}$. We define the zero velocity section Z as the three-dimensional space of zero velocity states $(\mathbf{q}, \mathbf{0})$.

The control system is written

$$(\dot{\mathbf{q}}, \ddot{\mathbf{q}}) = X_0(\mathbf{q}, \dot{\mathbf{q}}) + \sum_{i=1}^n u_i X_i(\mathbf{q}, \dot{\mathbf{q}}), \quad (1)$$

$(u_1, \dots, u_n)^T = \mathbf{u} \in U = \{\mathbf{0}, (1, 0, \dots, 0, 0)^T, (0, 1, \dots, 0, 0)^T, \dots, (0, 0, \dots, 1, 0)^T, (0, 0, \dots, 0, 1)^T\}$, where $X_0(\mathbf{q}, \dot{\mathbf{q}}) = (\dot{x}_w, \dot{y}_w, \dot{\theta}_w, 0, 0, 0)^T$ is the drift vector field, u_i is the thrust applied at the i th thruster, and $X_i(\mathbf{q}, \dot{\mathbf{q}})$ is the corresponding control vector field. The body \mathcal{B} has n thrusters. Only one thruster can be on at a time, and the thrust is unit. A feasible trajectory for \mathcal{B} is a solution of (1) for a control function $\mathbf{u}(t) \in U$ for all t .

To simplify the equations of motion, we choose the unit mass to be the mass of \mathcal{B} and the unit distance to be the radius of gyration of inertia of \mathcal{B} . Unit time is chosen to make the thrust magnitude unit. The control vector field X_i can then be written $(0, 0, 0, f_{xi} \cos \theta_w - f_{yi} \sin \theta_w, f_{xi} \sin \theta_w + f_{yi} \cos \theta_w, \tau_i)^T$, where (f_{xi}, f_{yi}) is the unit thruster force expressed in the frame \mathcal{F}_B ($f_{xi}^2 + f_{yi}^2 = 1$) and τ_i is the torque about the center of mass. We will write $\mathbf{f}_i = (f_{xi}, f_{yi}, \tau_i)^T$, where (f_{xi}, f_{yi}) is the linear component of \mathbf{f}_i .

Modifying notation from Nijmeijer and van der Schaft [8], we define $R^V(\mathbf{q}_0, \dot{\mathbf{q}}_0, T)$ to be the reachable set from $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$ at time $T > 0$ by feasible trajectories remaining in the neighborhood V of $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$ at times $t \in [0, T]$. Define $R^V(\mathbf{q}_0, \dot{\mathbf{q}}_0, \leq T) = \bigcup_{0 \leq t \leq T} R^V(\mathbf{q}_0, \dot{\mathbf{q}}_0, t)$. Then the system (1) (or simply the planar rigid body \mathcal{B}) is *small-time accessible* (or *locally accessible*) from $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$ if $R^V(\mathbf{q}_0, \dot{\mathbf{q}}_0, \leq T)$ contains a non-empty open set of $T\mathcal{C}$ for any neighborhood V of $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$ and all $T > 0$. \mathcal{B} is *small-time locally controllable* from $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$ if $R^V(\mathbf{q}_0, \dot{\mathbf{q}}_0, \leq T)$ contains a neighborhood of $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$ for any neighborhood V and all $T > 0$. \mathcal{B} is *controllable* from $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$ if, for any $(\mathbf{q}_1, \dot{\mathbf{q}}_1) \in T\mathcal{C}$, there exists a finite time T such that $(\mathbf{q}_1, \dot{\mathbf{q}}_1) \in R^{T\mathcal{C}}(\mathbf{q}_0, \dot{\mathbf{q}}_0, T)$. The phrase “from $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$ ” can be eliminated from each of these definitions if the condition applies at all $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$.

3 Previous Work

Partial controllability results for the planar rigid body with thrusters have been reported previously by Manikonda and Krishnaprasad [7] and Lewis and Murray [4]. Manikonda and Krishnaprasad [7] observed that the Hamiltonian dynamics of \mathcal{B} on the cotangent bundle $T^*\mathcal{C}$ are invariant to the Lie group \mathcal{C} , suggesting the study of the reduced dynamics on the three-dimensional

quotient manifold $T^*\mathcal{C}/\mathcal{C}$. They showed that the Lie-Poisson reduced dynamics of \mathcal{B} with a single bilateral thruster (or two opposing unilateral thrusters) are controllable on $T^*\mathcal{C}/\mathcal{C}$ provided the line of force does not pass through the center of mass— \mathcal{B} is controllable on its velocity space $T\mathcal{C}/\mathcal{C}$. They also showed that such a system is small-time accessible on the full state space $T\mathcal{C}$ but not small-time locally controllable.

Lewis and Murray [4] studied the set of reachable configurations for mechanical control systems starting at rest. For an initial configuration \mathbf{q} at zero velocity and neighborhood $V_{\mathcal{C}}$ of \mathbf{q} on \mathcal{C} , they define $R_{\mathcal{C}}^{V_{\mathcal{C}}}(\mathbf{q}, T)$ to be the set of reachable configurations (with any velocity) at time T by trajectories remaining in the configuration neighborhood $V_{\mathcal{C}}$. They call a system *small-time locally configuration controllable* if $R_{\mathcal{C}}^{V_{\mathcal{C}}}(\mathbf{q}, \leq T)$ contains a neighborhood of \mathbf{q} on \mathcal{C} for any neighborhood $V_{\mathcal{C}}$ and all $T > 0$. (This is a weaker condition than small-time local controllability, which requires the locally reachable set to be a neighborhood of $(\mathbf{q}, \mathbf{0})$ on the full state space, not just the configuration space.) They showed that a planar body with two bilateral thrusters (or four unilateral thrusters) is small-time locally configuration controllable. Lewis [3] also showed that a planar body with a single bilateral thruster is not small-time locally configuration controllable, and hence not small-time locally controllable.

Our interest in unilateral control forces arises from our previous work on robotic pushing of an object over a frictional support surface (Lynch and Mason [5]) and dynamic nonprehensile manipulation (Lynch and Mason [6]). In both cases, the robot controls the motion of an object by applying forces through unilateral contact. Unilateral inputs are fundamental to the controllability analyses of these systems, and the results of our work on dynamic manipulation lead directly to the results in this paper. Controllability with unilateral inputs has also been studied by Goodwine and Burdick [1].

The primary contribution of this paper is to generalize existing results to the case of unilateral thrusters and to provide new results regarding global controllability.

4 One Thruster

We begin by checking accessibility for a single thruster ($n = 1$). To test for small-time accessibility we examine the Lie algebra of the system vector fields. If \mathcal{V} is a family of vector fields (corresponding to constant controls) on a manifold M , then $L(\mathcal{V})$, the accessibility Lie algebra, is the smallest subalgebra of vector fields on M containing \mathcal{V} . (For a finite family \mathcal{V} , defining $B_0(\mathcal{V}) = \mathcal{V}$ and $B_{k+1}(\mathcal{V}) = B_k(\mathcal{V}) \cup \{[V_i, V_j] \text{ for all } V_i, V_j \in B_k(\mathcal{V})\}$, where the Lie bracket $[V_i, V_j]$ is given locally at $\mathbf{p} \in M$ as

$$[V_i, V_j](\mathbf{p}) = \frac{\partial V_j(\mathbf{p})}{\partial \mathbf{p}} V_i(\mathbf{p}) - \frac{\partial V_i(\mathbf{p})}{\partial \mathbf{p}} V_j(\mathbf{p}),$$

recall that the Lie algebra $L(\mathcal{V})$ is spanned by the elements of $B_{\infty}(\mathcal{V})$.) The tangent vectors of $L(\mathcal{V})$ at \mathbf{p} are $L(\mathcal{V})(\mathbf{p})$. Then the system satisfies the *Lie Algebra Rank Condition* at \mathbf{p} , and therefore is small-time accessible from \mathbf{p} , if $L(\mathcal{V})(\mathbf{p})$ is the tangent space $T_{\mathbf{p}}M$ (Hermann and Krener [2]; Sussmann [10]). Note that \mathcal{V} need not be symmetric for small-time accessibility; in particular, if V is an element of \mathcal{V} , it is not necessary that $-V$ also belong to \mathcal{V} .

For the case $n = 1$ we study the Lie algebra of the vector fields X_0 and X_1 . Without loss of generality, assume the thruster is aligned with the y -axis of $\mathcal{F}_{\mathcal{B}}$ ($\mathbf{f}_1 = (0, 1, \tau)^T$), yielding

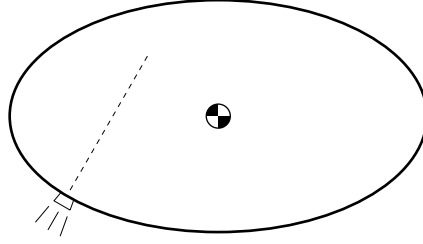


Figure 1: This planar body is small-time accessible because the single thruster provides nonzero torque about the center of mass.

$X_1 = (0, 0, 0, -\sin \theta_w, \cos \theta_w, \tau)^T$. We define the Lie bracket vector fields $X_2 = [X_0, X_1]$, $X_3 = [X_1, [X_0, X_1]]$, $X_4 = [X_1, [X_0, [X_0, X_1]]]$, $X_5 = [X_1, [X_1, [X_0, [X_0, X_1]]]]$, $X_6 = [X_0, [X_1, [X_1, [X_0, [X_0, X_1]]]]]$. We find that

$$\det(X_1 \ X_2 \ X_3 \ X_4 \ X_5 \ X_6) = -16\tau^8,$$

indicating that these six vector fields span the tangent space $T_{(\mathbf{q}, \dot{\mathbf{q}})}T\mathcal{C}$ at any state $(\mathbf{q}, \dot{\mathbf{q}})$, provided $\tau \neq 0$ (the line of action of the thruster must not pass through the center of mass).

Theorem 1 *The planar body \mathcal{B} is small-time accessible with a single unilateral thruster ($n = 1$) if and only if the line of action of the thruster does not pass through the center of mass.*

The single thruster of the planar body of Figure 1 provides nonzero torque, so the body is small-time accessible—for any neighborhood V of any state $(\mathbf{q}, \dot{\mathbf{q}})$, \mathcal{B} can reach a six-dimensional subset of $T\mathcal{C}$ without leaving V . Clearly $n = 1$ is never sufficient for controllability; the angular velocity of \mathcal{B} can change only in one direction.

5 Two Thrusters

Theorem 2 *The planar body \mathcal{B} is controllable with two unilateral thrusters ($n = 2$) if and only if the two thrusters provide torque of opposite signs ($\tau_1 > 0, \tau_2 < 0$).*

Proof: The conditions are clearly necessary. To prove they are sufficient, we need only prove

1. The planar body \mathcal{B} is controllable on its velocity space, the quotient space $T\mathcal{C}/\mathcal{C}$. Any velocity is attainable from any other velocity.
2. The planar body \mathcal{B} is controllable on its three-dimensional zero velocity section Z . \mathcal{B} can move from any $(\mathbf{q}_0, \mathbf{0})$ to any $(\mathbf{q}_1, \mathbf{0})$.

Using these properties, \mathcal{B} can be brought to a zero velocity state $(\mathbf{q}_1, \mathbf{0})$ (Property 1), moved to an arbitrary zero velocity state $(\mathbf{q}_2, \mathbf{0})$ (Property 2), and accelerated to the goal $(\mathbf{q}_3, \dot{\mathbf{q}}_3)$. This last step is possible because, by Property 1, there is a control $\mathbf{u}(\cdot)$ to take \mathcal{B} from zero velocity to the goal velocity $\dot{\mathbf{q}}_3$. Following the trajectory associated with $\mathbf{u}(\cdot)$ backward from $(\mathbf{q}_3, \dot{\mathbf{q}}_3)$ we obtain the zero velocity state $(\mathbf{q}_2, \mathbf{0})$.

Without loss of generality, let $\mathbf{f}_1 = (1, 0, \tau_1)^T$ and $\mathbf{f}_2 = (\cos \beta, \sin \beta, \tau_2)^T$, where $\tau_1 > 0, \tau_2 < 0$. (Simply define the x -axis of the frame \mathcal{F}_B to align with the linear component of \mathbf{f}_1 .) The angle of the linear component of \mathbf{f}_2 is β in \mathcal{F}_B .

To prove Theorem 2, we first state a technical lemma, then prove properties 1 and 2 above in Propositions 1 and 2.

Lemma 1 *Consider a body-fixed force \mathbf{f} , with a linear magnitude $f = \sqrt{f_x^2 + f_y^2} = 1$ and torque $\tau \neq 0$, applied to \mathcal{B} for a time t . Define the smooth function $m_{\mathbf{f}}(t, \omega_0)$ that maps the application time t and the initial angular velocity ω_0 of \mathcal{B} to the magnitude of the total linear impulse delivered during the application (linear forces integrated over t). Then for any range of initial angular velocities $(-\omega_{max}, \omega_{max}), \omega_{max} > 0$, there exists a time $T > 0$ such that $\partial m_{\mathbf{f}}(t, \omega_0)/\partial t > 0$ for $m_{\mathbf{f}}(t, \omega_0)$ restricted to $t \in (0, T), \omega_0 \in (-\omega_{max}, \omega_{max})$.*

Proof: The idea behind the lemma is simple. If \mathcal{B} does not rotate, the total linear impulse is ft . Rotation of \mathcal{B} , and therefore rotation of the force in the world frame \mathcal{F}_W , leads to some cancellation in the linear force components, giving a total linear impulse less than ft . If the rotation of \mathcal{B} during the application is less than $\pi/2$, however, clearly $m_{\mathbf{f}}(t, \omega_0)$ is monotonic with t (the cosine of the angle between any two instantaneous forces is positive). The magnitude of the angular velocity during the force application is upper-bounded by $|\omega_0| + T|\tau|$, so there always exists a $T > 0$ found by

$$(|\omega_0| + T|\tau|)T = \frac{\pi}{2} \Rightarrow T = \frac{-|\omega_0| + \sqrt{|\omega_0|^2 + 2\pi|\tau|}}{2|\tau|}$$

such that $\partial m_{\mathbf{f}}(t, \omega_0)/\partial t > 0$ for all $t \in (0, T)$.

More formally, from a state $(0, 0, 0, v_x, v_y, \omega_0)^T$, the equations of motion during the application of a force $(f, 0, \tau)^T$ are given by:

$$\begin{aligned} \ddot{\theta}_w(t) &= \tau \\ \ddot{x}_w(t) &= f \cos \theta_w(t) = f \cos\left(\frac{\tau t^2}{2} + \omega_0 t\right) \\ \ddot{y}_w(t) &= f \sin \theta_w(t) = f \sin\left(\frac{\tau t^2}{2} + \omega_0 t\right) \end{aligned}$$

Integrating, we obtain expressions for $\dot{x}_w(t)$ and $\dot{y}_w(t)$ in terms of Fresnel integrals, and $m_{\mathbf{f}}(t, \omega_0) = ((\dot{x}_w(t) - \dot{x}_w(0))^2 + (\dot{y}_w(t) - \dot{y}_w(0))^2)^{\frac{1}{2}}$. We find that $\partial m_{\mathbf{f}}(t, \omega_0)/\partial t \rightarrow f$ as t approaches 0 from above, for any ω_0 , verifying the lemma for any given $(-\omega_{max}, \omega_{max})$. An example plot of $m_{\mathbf{f}}(t, \omega_0)$ for $f = \tau = 1$ is shown in Figure 2. \square

We now prove properties 1 and 2 above.

Proposition 1 *A planar body \mathcal{B} with two thrusters, $\mathbf{f}_1, \mathbf{f}_2$ with $\tau_1 > 0, \tau_2 < 0$, is controllable on its velocity space, the quotient space TC/\mathcal{C} . Any velocity is attainable from any other velocity.*

Proof: We decouple the proof of this property by showing that the body's angular velocity can be changed arbitrarily without changing the linear velocity, and the linear velocity can be changed arbitrarily without changing the angular velocity. We begin with the former.

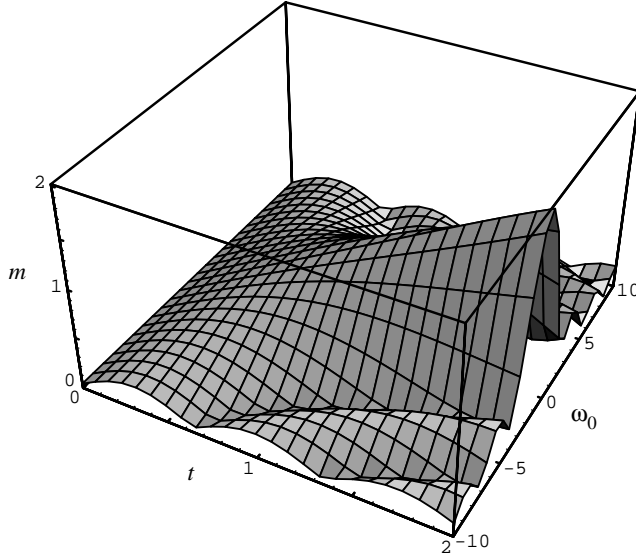


Figure 2: The total linear impulse m delivered to the planar body \mathcal{B} as a function of its initial angular velocity ω_0 and the time of application t of the force $(f_x, f_y, 1)^T$, where $f_x^2 + f_y^2 = 1$.

There exists a $\Delta\omega > 0$, for any initial angular velocity $\dot{\theta}_w$, such that two applications of the force \mathbf{f}_1 (respectively \mathbf{f}_2), for times t_1 and t_2 , can change the angular velocity to any value in the open interval $(\dot{\theta}_w, \dot{\theta}_w + \Delta\omega)$ (resp. $(\dot{\theta}_w - \Delta\omega, \dot{\theta}_w)$) without changing the linear velocity. By patching together open sets, \mathcal{B} can be moved from $(\dot{x}_{w1}, \dot{y}_{w1}, \dot{\theta}_{w1})$ to $(\dot{x}_{w1}, \dot{y}_{w1}, \dot{\theta}_{w2})$ for any $\dot{x}_{w1}, \dot{y}_{w1}, \dot{\theta}_{w1}, \dot{\theta}_{w2}$.

To see this, consider \mathbf{f}_1 (similar reasoning holds for \mathbf{f}_2) applied for a time t_1 at an initial angular velocity ω_1 . By Lemma 1, there exists a neighborhood W of $(0, \omega_1)$ such that for all $(t, \omega) \in W$, $\partial m_{\mathbf{f}_1}(t, \omega)/\partial t > 0$ ($m_{\mathbf{f}_1}$ does not achieve a local maximum or minimum, and the constant $m_{\mathbf{f}_1}$ contours never become parallel to the t -axis). Therefore we can choose t_1 from an open interval $(0, T)$ such that $(t_1, \omega_1) \in W$ and there exists a $(t_2, \omega_1 + t_1\tau_1) \in W$ (where $\omega_1 + t_1\tau_1$ is the angular velocity after application of \mathbf{f}_1 for time t_1) such that $m_{\mathbf{f}_1}(t_1, \omega_1) = m_{\mathbf{f}_1}(t_2, \omega_1 + t_1\tau_1)$. In other words, the total linear impulse delivered by the first application can be exactly matched by the linear impulse of the second application. By allowing \mathcal{B} to rotate sufficiently between applications, the linear impulses cancel, restoring the original linear velocity, while the angular velocity is transferred to a point of an open interval $(\omega_1, \omega_1 + \Delta\omega)$ parameterized by $t_1 \in (0, T)$.

Note that if t_1 is chosen to exactly zero the angular velocity after the first application ($t_1 = -\omega_1/\tau_1$, $\omega_1 < 0$), \mathcal{B} cannot rotate to the angle for the second application. We simply avoid such a value of t_1 , possibly creating the two open t_1 intervals $(0, -\omega_1/\tau_1)$ and $(-\omega_1/\tau_1, T)$.

To see that the linear velocity can be changed arbitrarily without changing the angular velocity, first assume \mathcal{B} is always rotating. The forces \mathbf{f}_1 and \mathbf{f}_2 can be applied in pairs such that their total torque impulses cancel and their linear components yield a change of velocity in the desired direction. Because \mathcal{B} is rotating, the linear components of \mathbf{f}_1 and \mathbf{f}_2 can take any direction in $\mathcal{F}_{\mathcal{W}}$.

Finally, \mathcal{B} can be maneuvered to any desired velocity by first transferring it to the desired linear velocity (possibly after giving it an initial angular velocity) and then transferring it to the desired angular velocity. \square

Proposition 2 *A planar body \mathcal{B} with two thrusters, $\mathbf{f}_1, \mathbf{f}_2$ with $\tau_1 > 0, \tau_2 < 0$, is controllable on its three-dimensional zero velocity section Z . \mathcal{B} can move from any $(\mathbf{q}_0, \mathbf{0})$ to any $(\mathbf{q}_1, \mathbf{0})$.*

Proof: This can be proven by demonstrating that \mathbf{f}_1 and \mathbf{f}_2 are sufficient to steer \mathcal{B} from any zero velocity state $(\mathbf{q}, \mathbf{0})$ to a neighborhood of \mathbf{q} on the three-dimensional zero velocity section Z . By patching together neighborhoods, \mathcal{B} can be moved from any $(\mathbf{q}_0, \mathbf{0})$ to any $(\mathbf{q}_1, \mathbf{0})$. (Construct any curve in Z connecting these two states. Each point on the curve is in the interior of its open accessible set. These open sets comprise an open cover of the curve, and because the curve is compact, there is a finite subcover.)

To greatly simplify the discussion, we first consider the limiting case where control forces are applied in short bursts (impulses) such that the motion of \mathcal{B} during the application of a control force is zero. We then relax this assumption. The angular impulses delivered during applications of \mathbf{f}_1 and \mathbf{f}_2 are ϵ and $-\epsilon$, respectively, where ϵ is a small positive value. The magnitudes of the linear impulses are $k_1\epsilon$ and $k_2\epsilon$, respectively, where $k_1 = 1/\tau_1, k_2 = 1/|\tau_2|$. Consider the following four-step sequence beginning with \mathcal{B} at $(\mathbf{0}, \mathbf{0})$:

1. Apply an \mathbf{f}_1 impulse and allow \mathcal{B} to drift for time π/ϵ . The new state is $(k_1\pi, 0, \pi, k_1\epsilon, 0, \epsilon)^T$.
2. Apply an \mathbf{f}_1 impulse, canceling \mathcal{B} 's linear velocity and doubling its angular velocity. Allow \mathcal{B} to rotate in place an angle ψ_1 , where ψ_1 can be chosen arbitrarily. The new state is $(k_1\pi, 0, \pi + \psi_1, 0, 0, 2\epsilon)^T$.
3. Apply an \mathbf{f}_2 impulse and allow \mathcal{B} to drift for time π/ϵ . The new state is $(\pi k_1 + \pi k_2 \cos(\pi + \psi_1 + \beta), \pi k_2 \sin(\pi + \psi_1 + \beta), 2\pi + \psi_1, k_2\epsilon \cos(\pi + \psi_1 + \beta), k_2\epsilon \sin(\pi + \psi_1 + \beta), \epsilon)^T$.
4. Apply an \mathbf{f}_2 impulse, stopping \mathcal{B} 's motion. The final state is $(\pi k_1 + \pi k_2 \cos(\pi + \psi_1 + \beta), \pi k_2 \sin(\pi + \psi_1 + \beta), 2\pi + \psi_1, 0, 0, 0)^T$.

The final set of configurations R_1 reachable from the zero configuration $\mathbf{0}$ is a one-dimensional curve on Z , parameterized by ψ_1 , given by $f(\psi_1, \mathbf{0}) = (\pi(k_1 - k_2 \cos(\psi_1 + \beta)), -\pi k_2 \sin(\psi_1 + \beta), \psi_1)^T$. The mapping f is independent of ϵ ; ϵ determines only the time of motion. We consider ψ_1 to be the control. Repeating the sequence from each point of R_1 , we obtain a reachable set R_{12} , and repeating again from each point of R_{12} , we obtain a set of reachable configurations $R_{123} = \{f(\psi_3, f(\psi_2, f(\psi_1, \mathbf{0}))) | \psi_1, \psi_2, \psi_3 \in S^1\}$ on Z :

$$\begin{aligned} & (\pi(k_1 - k_2 \cos(\psi_1 + \beta)) + \cos \psi_1(k_1 - k_2 \cos(\psi_2 + \beta)) + k_2 \sin \psi_1 \sin(\psi_2 + \beta) \\ & \quad + \cos(\psi_1 + \psi_2)(k_1 - k_2 \cos(\psi_3 + \beta)) + k_2 \sin(\psi_1 + \psi_2) \sin(\psi_3 + \beta)), \\ & \pi(-k_2 \sin(\psi_1 + \beta) + \sin \psi_1(k_1 - k_2 \cos(\psi_2 + \beta)) - k_2 \cos \psi_1 \sin(\psi_2 + \beta) \\ & \quad + \sin(\psi_1 + \psi_2)(k_1 - k_2 \cos(\psi_3 + \beta)) - k_2 \cos(\psi_1 + \psi_2) \sin(\psi_3 + \beta)), \\ & \quad \psi_1 + \psi_2 + \psi_3)^T. \end{aligned}$$

(Note that angles are modulo 2π .)

We wish to show that this set contains a neighborhood of the origin on Z . First we observe that by choosing $\psi_1 = \psi_2 = \psi_3 = 2\pi/3$, \mathcal{B} returns to the zero configuration $(0, 0, 0)^T$. We will

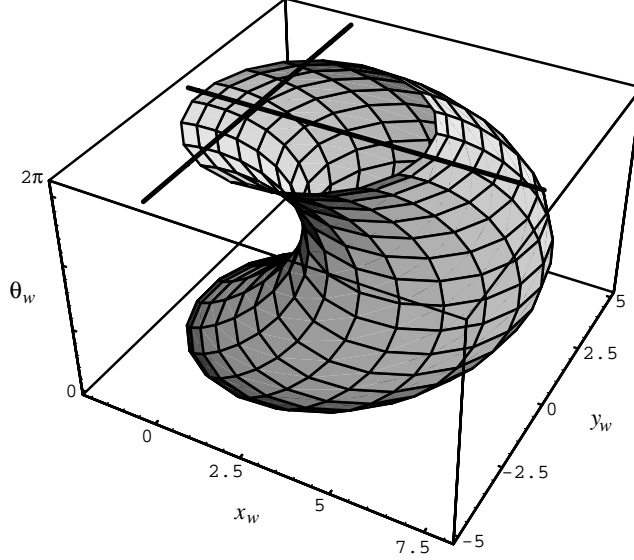


Figure 3: The reachable set R_{123} for $\mathbf{f}_1 = (1, 0, 1)^T$, $\mathbf{f}_2 = (1, 0, -2)^T$. Note that the bent tube wraps around at $\theta_w = 2\pi$ and that $(0, 0, 0)^T$ is in the interior.

define this point in the three-dimensional control space as $\Psi_0 = (2\pi/3, 2\pi/3, 2\pi/3)^T$. To show that $(0, 0, 0)^T$ is in the interior of R_{123} on Z , we look at the determinant of the Jacobian of R_{123}

$$\det \begin{pmatrix} \frac{\partial x_w}{\partial \psi_1} & \frac{\partial x_w}{\partial \psi_2} & \frac{\partial x_w}{\partial \psi_3} \\ \frac{\partial y_w}{\partial \psi_1} & \frac{\partial y_w}{\partial \psi_2} & \frac{\partial y_w}{\partial \psi_3} \\ \frac{\partial \theta_w}{\partial \psi_1} & \frac{\partial \theta_w}{\partial \psi_2} & \frac{\partial \theta_w}{\partial \psi_3} \end{pmatrix} = \pi^2(k_1^2 + k_2^2 - 2k_1k_2 \cos \beta) \sin \psi_2.$$

The determinant is nonzero at Ψ_0 unless $k_1^2 + k_2^2 - 2k_1k_2 \cos \beta = 0$. This equation holds only when $k_1 = k_2 = 1/\tau_1$, $\beta = 0$, i.e., $\mathbf{f}_2 = (1, 0, -\tau_1)^T$. For any other choice of \mathbf{f}_2 , the set R_{123} gives a neighborhood of reachable configurations of $(0, 0, 0)^T$ on Z . Figure 3 shows the reachable set R_{123} for $\mathbf{f}_1 = (1, 0, 1)^T$, $\mathbf{f}_2 = (1, 0, -2)^T$.

If $\mathbf{f}_2 = (1, 0, -\tau_1)^T$, the reachable configuration space by the control sequence above is a one-dimensional curve, regardless of the number of applications of the sequence. Therefore we modify the control sequence above by prepending it with the following two steps (assume \mathcal{B} begins at $(\mathbf{0}, \mathbf{0})$, and $k = 1/\tau_1$):

- Apply an \mathbf{f}_1 impulse and allow \mathcal{B} to drift for time π/ϵ . The new state is $(\pi k, 0, \pi, k\epsilon, 0, \epsilon)^T$.
- Apply an \mathbf{f}_2 impulse, stopping \mathcal{B} 's motion. The new state is $(\pi k, 0, \pi, 0, 0, 0)^T$.

Then continuing with the four steps described previously, the final reachable curve is $f(\psi_1, \mathbf{0}) = (\pi k \cos \psi_1, \pi k \sin \psi_1, \psi_1 + \pi)^T$. Applying the complete six-step control sequence two more times, we get $R_{123} = \{f(\psi_3, f(\psi_2, f(\psi_1, \mathbf{0}))) | \psi_1, \psi_2, \psi_3 \in S^1\}$ on Z :

$$\begin{aligned} &(\pi k(\cos \psi_1(1 - \cos \psi_2) + \sin \psi_1 \sin \psi_2 + \cos(\psi_1 + \psi_2) \cos \psi_3 - \sin(\psi_1 + \psi_2) \sin \psi_3), \\ &\pi k(\sin \psi_1(1 - \cos \psi_2) - \cos \psi_1 \sin \psi_2 + \sin(\psi_1 + \psi_2) \cos \psi_3 + \cos(\psi_1 + \psi_2) \sin \psi_3), \\ &\psi_1 + \psi_2 + \psi_3 + \pi)^T. \end{aligned}$$

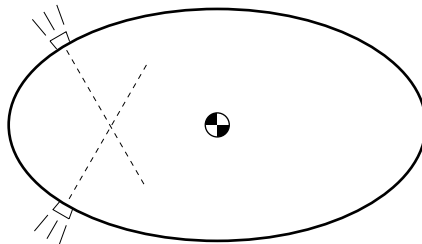


Figure 4: This planar body is controllable because the two thrusters provide opposing torques.

The control $\Psi_0 = (5\pi/3, 5\pi/3, 5\pi/3)^T$ returns \mathcal{B} to $(0, 0, 0)^T$. The determinant of the Jacobian is $-\pi^2 k^2 \sin \psi_2$, indicating that this control is nonsingular— R_{123} contains a neighborhood of $(0, 0, 0)^T$ on Z .

We have shown that by applying controls in a neighborhood V of Ψ_0 in the control space, \mathcal{B} , starting from the zero velocity configuration $(0, 0, 0)^T$, can reach a neighborhood W of $(0, 0, 0)^T$ on Z using impulses. W is invariant to \mathbf{q} when expressed in the frame $\mathcal{F}_{\mathcal{B}}$, so from any configuration \mathbf{q} on Z , \mathcal{B} can reach a neighborhood of configurations of \mathbf{q} on Z .

Finally we replace the impulses with the control forces $\mathbf{f}_1, \mathbf{f}_2$. Consider the first two steps of the four-step control sequence. The impulse of the first step is approximated by an application of \mathbf{f}_1 for a time $T/2$ (linear impulse is $m_1 = m_{\mathbf{f}_1}(T/2, 0)$), free rotation of \mathcal{B} by an angle δ , and another application for a time $T/2$ (linear impulse is $m_2 = m_{\mathbf{f}_1}(T/2, T\tau_1/2)$). The second step is approximated by an application of \mathbf{f}_1 for a time T (linear impulse is $m_3 = m_{\mathbf{f}_1}(T, T\tau_1)$). The total torque impulses of the two steps are equal. For small values of T , δ can be chosen so that the total linear impulse of the two steps are equal, thus zeroing the linear velocity after the two steps. (Note that $m_1 + m_2 > m_3 > m_1 > m_2$, and δ allows partial cancellation between m_1 and m_2 . As $T \rightarrow 0$, $\delta \rightarrow 0$.) Thus the final velocity is $(0, 0, 2T\tau_1)^T$, identical to the impulse case where $\epsilon = T\tau_1$. The configuration error vector (due to motion of \mathcal{B} during the force application and the linear velocity error) is $e_1(\psi_1, T)$, smooth in ψ_1 and T , and $e_1(\psi_1, T) \rightarrow \mathbf{0}$ as $T \rightarrow 0$.

Continuing in an analogous manner for steps 3 and 4 (\mathcal{B} returns to zero velocity after step 4), and repeating the sequence twice more, we get a smooth final configuration error $e_{123}(\psi_1, \psi_2, \psi_3, T)$ such that $e_{123}(\psi_1, \psi_2, \psi_3, T) \rightarrow \mathbf{0}$ as $T \rightarrow 0$. For any $W \subset R_{123}$ above, where W is a neighborhood of $(0, 0, 0)^T$ on Z , we can define the set $W' = \{w + e_{123}(\psi_1, \psi_2, \psi_3, T) | w \in W\}$, where ψ_1, ψ_2, ψ_3 are the controls (for the impulse case) that take \mathcal{B} to w . By choosing T small enough, W' also contains a neighborhood of $(0, 0, 0)^T$ on Z . Therefore \mathcal{B} can reach a neighborhood of any initial configuration \mathbf{q} on Z with \mathbf{f}_1 and \mathbf{f}_2 , and \mathcal{B} is controllable on its zero velocity section. \square

This completes the proof of Theorem 2. \square

The planar body \mathcal{B} of Figure 4 is controllable by Theorem 2.

Finally, we note that two thrusters are never sufficient for small-time local controllability. Lewis [3] and Manikonda and Krishnaprasad [7] showed that a bilateral thruster (or two opposing unilateral thrusters) are insufficient for small-time local controllability at the origin. If the two thrusters are not opposing, then the forces \mathbf{f}_1 and \mathbf{f}_2 lie in an open half-space of the body-fixed force space (f_x, f_y, τ) . Therefore the tangent vectors $X_1(\mathbf{q}, \dot{\mathbf{q}})$ and $X_2(\mathbf{q}, \dot{\mathbf{q}})$ are confined to an open half-space of $T_{(\mathbf{q}, \dot{\mathbf{q}})}TC$ at any state $(\mathbf{q}, \dot{\mathbf{q}})$, and \mathcal{B} is not small-time locally controllable (Sussmann [9]).

6 Three Thrusters

Under certain conditions, the Lie Algebra Rank Condition can be used to prove small-time local controllability. One such condition is that the system vector fields are *symmetric*—all vector fields can be followed forward and backward. Sussmann [10] generalized this notion of input symmetries to prove a general sufficient condition for small-time local controllability. We use a specific version of his result to prove that a planar body with three thrusters can be small-time locally controllable on the zero velocity section Z .

Theorem 3 *The planar body \mathcal{B} is small-time locally controllable on the zero velocity section Z with three unilateral thrusters ($n = 3$) if their lines of action intersect at a single point (which is not the center of mass) and the linear components of $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ positively span the plane.*

Remark: A set of k vectors $\mathbf{r}_1, \dots, \mathbf{r}_k \in \mathbf{R}^m$ positively spans \mathbf{R}^m if and only if for any $\mathbf{r} \in \mathbf{R}^m$ there exists a set of nonnegative weights $\lambda_1, \dots, \lambda_k \geq 0$ such that $\sum_{i=1}^k \lambda_i \mathbf{r}_i = \mathbf{r}$. (Equivalently, the convex hull of the vectors \mathbf{r}_i contains the origin in its interior.) In Theorem 3 we have $m = 2$, $k = 3$, and $\mathbf{r}_i = (f_{xi}, f_{yi})$.

Proof: Consider the system (Lewis and Murray [4])

$$(\dot{\mathbf{q}}, \ddot{\mathbf{q}}) = X_0(\mathbf{q}, \dot{\mathbf{q}}) + u_1 X_1(\mathbf{q}, \dot{\mathbf{q}}) + u_2 X_2(\mathbf{q}, \dot{\mathbf{q}}), \quad |u_i| \leq 1. \quad (2)$$

We will also need the bracket terms $X_3 = [X_0, X_1]$, $X_4 = [X_0, X_2]$, $X_5 = [X_1, [X_0, X_2]]$, $X_6 = [X_0, [X_1, [X_0, X_2]]]$. We show that small-time local controllability of (2) with two bilateral thrusters implies small-time local controllability of (1) with three unilateral thrusters satisfying the conditions of Theorem 3.

Now we give some definitions necessary to apply a version of Sussmann’s [10] sufficient condition for small-time local controllability to a system such as (2). For a bracket term B , we define $\delta_i(B)$ as the number of times X_i appears in B , and the degree of B is $\sum_{i=0}^n \delta_i(B)$. B is called a “bad” bracket if $\delta_0(B)$ is odd and $\delta_i(B)$ is even for all $i \in \{1, \dots, n\}$, and B is a “good” bracket otherwise. A “bad” bracket B is “neutralized” at a state \mathbf{p} if B , evaluated at \mathbf{p} , is the linear combination of “good” brackets of lower degree evaluated at \mathbf{p} . Sussmann proved that if the system satisfies the Lie Algebra Rank Condition at \mathbf{p} and all “bad” brackets evaluated at \mathbf{p} are neutralized, then the system is small-time locally controllable at \mathbf{p} .

Consider the control vector fields $X_1 = (0, 0, 0, \cos \theta_w, \sin \theta_w, 0)^T$ and $X_2 = (0, 0, 0, -\sin \theta_w, \cos \theta_w, \tau)^T$. The force \mathbf{f}_1 acts through the center of mass along the x -axis of $\mathcal{F}_{\mathcal{B}}$ and \mathbf{f}_2 acts along the y -axis with torque τ about the center of mass. The force lines intersect at a point C not at the center of mass. Calculating the brackets above, we find that $\det(X_1 \ X_2 \ X_3 \ X_4 \ X_5 \ X_6) = \tau^4$; the Lie Algebra Rank Condition is satisfied provided $\tau \neq 0$. Because we only use brackets up to degree four, the only “bad” brackets to be neutralized are the drift field (which vanishes at $\dot{\mathbf{q}} = \mathbf{0}$) and the “bad” brackets of degree three $[X_1, [X_0, X_1]]$ and $[X_2, [X_0, X_2]]$. We have

$$[X_1, [X_0, X_1]] = (0, 0, 0, 0, 0, 0)^T$$

$$[X_2, [X_0, X_2]] = (0, 0, 0, -2\tau \cos \theta_w, -2\tau \sin \theta_w, 0)^T$$

which are clearly neutralized (the latter being a multiple of X_1), and the system is small-time locally controllable. The two forces \mathbf{f}_1 and \mathbf{f}_2 span a force/torque plane \mathcal{P} which is not the $\tau = 0$

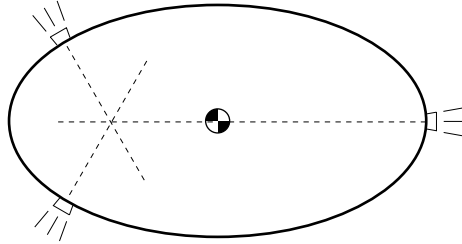


Figure 5: This planar body is small-time locally controllable at any state $(\mathbf{q}, \mathbf{0})$ by the three unilateral thrusters shown.

plane, and the controls $|u_i| \leq 1$ define a compact, convex subset of \mathcal{P} containing the origin in the interior (relative to \mathcal{P}). By Sussmann’s [10] Proposition 2.3, small-time local controllability for this system implies small-time local controllability for the bang-bang system with the extremal controls $u_1, u_2 \in \{-1, 1\}$. Scaling, small-time local controllability holds for any compact, convex set of control forces that contains a neighborhood of the origin in \mathcal{P} , and Sussmann’s proposition indicates that the extremal forces alone are sufficient. Therefore, any set of control forces that *positively* spans the plane \mathcal{P} also yields small-time local controllability. Any three unilateral forces which intersect at C and positively span the (x, y) plane also positively span \mathcal{P} . \square

A planar body which is small-time locally controllable on Z is shown in Figure 5. This planar body can follow any path on Z arbitrarily closely.

7 Rotating Thrusters

Now suppose the planar body has a single unilateral thruster which can rotate in place with respect to the body (Manikonda and Krishnaprasad [7]). The thruster is not located at the center of mass, and the x -axis of \mathcal{F}_B is aligned with the line through the thruster and the center of mass (Figure 6). The thruster has two controls: the thrust value $u \in \{0, 1\}$, and the rotation angle of the thruster relative to \mathcal{F}_B , $\phi \in \Phi = [\phi_{min}, \phi_{max}]$. The Lebesgue measure of the angle interval Φ is denoted $|\Phi|$. Then Theorem 2 can be used to show that the body is controllable provided $0 \in (\phi_{min}, \phi_{max})$ or $\pi \in (\phi_{min}, \phi_{max})$ — Φ contains a neighborhood of 0 or π (remembering that angles are modulo 2π). This implies that the thruster can apply both positive and negative torques. Similarly, Theorem 3 can be used to show that the body is small-time locally controllable on Z if $|\Phi| > \pi$. Such a thruster can apply forces which positively span the plane and pass through a single point in \mathcal{F}_B (the thruster position) not at the center of mass.

8 Conclusion

This note has derived the minimum number of unilateral thrusters necessary for three different versions of nonlinear controllability of a planar rigid body. Challenging open problems include finding optimal controls for controllable planar bodies with two unilateral thrusters or small-time locally controllable planar bodies with three unilateral thrusters, and finding nonsmooth or time-varying feedback controllers.

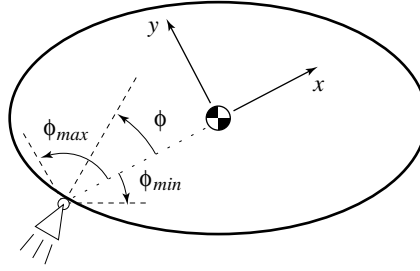


Figure 6: Notation for a planar body with a rotating thruster.

Acknowledgments

This note grew directly out of joint work with Matt Mason. I thank Yan-bin Jia, P. S. Krishnaprasad, Andrew Lewis, and Richard Murray for discussions regarding this work.

References

- [1] B. Goodwine and J. Burdick. Controllability with unilateral control inputs. In *Conference on Decision and Control*, pages 3394–3399, 1996.
- [2] R. Hermann and A. J. Krener. Nonlinear controllability and observability. *IEEE Transactions on Automatic Control*, AC-22(5):728–740, Oct. 1977.
- [3] A. D. Lewis. Local configuration controllability for a class of mechanical systems with a single input. *1997 European Control Conference*.
- [4] A. D. Lewis and R. M. Murray. Configuration controllability of simple mechanical control systems. *SIAM Journal on Control and Optimization*, 35(3):766–790, May 1997.
- [5] K. M. Lynch and M. T. Mason. Stable pushing: Mechanics, controllability, and planning. *International Journal of Robotics Research*, 15(6):533–556, Dec. 1996.
- [6] K. M. Lynch and M. T. Mason. Dynamic nonprehensile manipulation: Controllability, planning, and experiments, 1998. *International Journal of Robotics Research*, to appear.
- [7] V. Manikonda and P. S. Krishnaprasad. Controllability of Lie-Poisson reduced dynamics. Institute for Systems Research 57-59, University of Maryland, May 1997.
- [8] H. Nijmeijer and A. J. van der Schaft. *Nonlinear Dynamical Control Systems*. Springer-Verlag, 1990.
- [9] H. J. Sussmann. A sufficient condition for local controllability. *SIAM Journal on Control and Optimization*, 16(5):790–802, Sept. 1978.
- [10] H. J. Sussmann. A general theorem on local controllability. *SIAM Journal on Control and Optimization*, 25(1):158–194, Jan. 1987.