Multi-Agent Coordination by Decentralized Estimation and Control

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***Abstract*— We describe a framework for the design of collective behaviors for groups of identical mobile agents. The approach is based on decentralized simultaneous estimation and control, where each agent communicates with neighbors and estimates the global performance properties of the swarm needed to make a local control decision. Challenges of the approach include designing a control law with desired convergence properties, assuming each agent has perfect global knowledge; designing an estimator that allows each agent to make correct estimates of the global properties needed to implement the controller; and possibly modifying the controller to recover desired convergence properties when using the estimates of global performance. We apply this framework to two different problems: (1) controlling the moment statistics describing the location and shape of a swarm, and (2) cooperative target localization. For the swarm formation control problem, we derive small-gain conditions which, if satisfied, guarantee that the formation statistics are driven to desired values, even in the presence of a changing network topology and the addition and deletion of robots.**

***Index Terms*— Multi-agent systems, decentralized control, distributed control, dynamic average consensus estimation, formation control.**

I. Introduction

We are studying the following general problem: given a set of identical mobile agents, design a control law to run on each agent, based on sensor and communication input, to achieve a desired collective "emergent" global behavior of the system. In other words, the global dynamical system defined by the interaction of the many individual agents' control laws should have the desired collective behavior as an attractor, preferably a global attractor. The performance of the system is judged by the global behavior of the system—it must be evaluated over all the agents.

The key constraints are that each agent is identical, has limited sensing, computation, motion, and communication capabilities, and may have significant dynamics. The performance of the group should improve or degrade gracefully as agents are added or deleted; in other words, the approach should be scalable, robust, and require no central controller.

While intelligent collective behavior can emerge even when each agent is ignorant of global properties of the system, as demonstrated in a number of models of schooling and flocking [9], [35], [41], [43], there are few tools to guide the design of local control laws to achieve a particular desired collective behavior. Only a limited class of tasks can be solved by reactive controllers using immediate sensory and communication input from neighbors, and some of these require that the agents be able to implement a sensing or communication network of a particular structure. Instead, we are pursuing a design framework for a broader class of tasks, with changing numbers of agents and few requirements on the time-varying communication network. We begin with the desired behavior encoded in a global cost functional J. We then equip each agent with (a) a local controller that chooses an action to minimize the cost based on knowledge of the global performance of the group, and (b) an estimator that provides estimates of the global properties of the group needed to implement the controller. This estimator uses sensory data and information communicated by other agents.

To ensure desired convergence properties when using estimates, we can modify the controller according to the properties of the estimator. For example, we can enforce a small-gain condition or a time-scale separation of the estimator and controller dynamics. These conditions can typically be written as bounds on motion control gains as a function of the estimator's communication network structure. Deriving such conditions is one of the main goals of our work. In Section III, for example, we provide small-gain conditions that express bounds on the aggressiveness of the motion controls as a function of the communication graph Laplacian.

The estimate-and-control approach can be applied to both motion coordination tasks and mobile sensing tasks. This is illustrated by the examples of *formation control*, where the cost function is in terms of the positions of the agents, and *cooperative target localization*, where the cost function is in terms of the uncertainty in the agents' estimates of the moving target. In the formation control problem, the agents' global estimates serve the motion coordination problem; in the target localization problem, motion control serves to improve the agents' estimates.

Example 1 (Formation Control): We can encode an abstraction of a robot formation using a finite number of geometric moments [2]. If the position of agent i in a plane is denoted $p_i = [p_{ix}\ p_{iy}]^T$, then a formation of n robots can be abstracted to an ℓ-vector of moments of the form

$$f(p) = \frac{1}{n}\sum_{i=1}^{n}[p_{ix}\ \ p_{iy}\ \ p_{ix}^2\ \ p_{iy}^2\ \ p_{ix}p_{iy}\ \ p_{ix}^3\ \ \ldots]^T\,,$$

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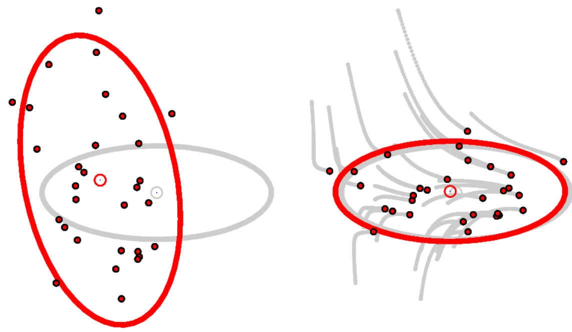


Fig. 1. (Left) The initial configuration of a swarm, a uniform-density ellipse with the same mass and first- and second-order moments as the swarm, and the goal formation of the swarm represented as another uniform-density ellipse. (Right) The swarm converges to a configuration having the desired first- and second-order moment statistics.

where $p = [p_1^T \dots p_n^T]^T$. To achieve a desired vector of moments f^*, which may be commanded by an exogenous input or derived from environmental variables, we can define a cost function

$$J = [f(p) - f^*]^T \Gamma [f(p) - f^*] \,,$$

where Γ is a positive-definite weighting matrix. This is the primary example of the paper, studied in detail in Section III (see Figure 1).

Example 2 (Cooperative Target Localization): A group of agents cooperatively tracks the location of a target in the environment. Each agent takes noisy sensor range and bearing readings of the target and maintains an estimate of the target location. Each agent shares its estimate with its neighbors to reach consensus on a belief distribution of the location of the target, represented by a mean and a covariance matrix P. The noise in the information gathered by each agent's sensor is dependent on the relative location of the target, so each agent moves to maximize the expected new sensor information relative to the current uncertainty P. The cost function can be expressed as

$$J = \det(P) \,.$$

The bulk of this paper presents the estimate-and-control approach applied to the formation control problem. Our primary interest is in large swarms where it is not necessary to specify the exact position of each robot. Instead, the swarm formation can be described by a set of summary moment statistics that form a basis for the space of all formations. These statistics have the property that low-order moments capture much of the essential shape of the swarm, but progressively higher-order moments can be specified until only a single formation is consistent with the statistics. Low-order moments then provide a convenient abstraction of the total swarm formation, allowing, for example, high-level human interaction with a large number of robots. We show that the estimate-and-control approach provides guarantees on the convergence of the swarm to desired formation statistics in the face of changing communication networks and the addition and deletion of agents.

Section II outlines the general framework and places it in the context of previous work in multi-agent control. Section III contains the main results of the paper in the form of estimator and motion control algorithms for the formation control problem. Proofs of the correctness of these algorithms are deferred to the Appendix. Section IV briefly outlines the cooperative target localization problem and provides simulations demonstrating the performance. Section V outlines a number of directions for future extension of the results.

II. Framework

The system consists of a collection of n identical mobile agents, each implementing the same control system with no identifying tags or ID numbers. Agents may be added or deleted from the system at any time, i.e., n may change. The (physical) state of agent i is written as the vector x_i, its control is the vector u_i, and its state evolves according to the dynamics

$$\dot{x}_i = F(x_i, u_i, \mathcal{X}_i^{\text{phys}}) \,, \tag{1}$$

where $\mathcal{X}_i^{\text{phys}}$ represents the set of states of any nearby agents that may physically impact the motion of agent i (e.g., through contact, fluid wake, hydrophobic effects, etc.). For the problems we consider, $\mathcal{X}_i^{\text{phys}}$ can be omitted from the motion dynamics, yielding $\dot{x}_i = F(x_i, u_i)$.

Each agent takes measurements of the form

$$z_i = C(x_i, \mathcal{X}_i^{\text{sens}}) \,, \tag{2}$$

where $\mathcal{X}_i^{\text{sens}}$ represents the states of other agents that affect or contribute to the measurements. The vector z_i may measure absolute or relative positions and velocities of the agents, or environmental variables such as temperature, salinity, or locations of points of interest in the environment. This vector also includes any exogenous reference inputs. The dimension of z_i may change depending on the number of neighbors in the sensing network.

We are interested in the design of dynamic local state feedback controllers of the form[1]

$$u_i = K(x_i, z_i, \eta_i, y_i, \mathcal{S}_i) \tag{3}$$
$$s_i = G(x_i, z_i, \eta_i, y_i, \mathcal{S}_i) \tag{4}$$
$$\dot{\eta}_i = Q(x_i, z_i, \eta_i, y_i, \mathcal{S}_i) \tag{5}$$
$$y_i = R(x_i, z_i, \eta_i, \mathcal{S}_i) \,, \tag{6}$$

where s_i denotes the fixed-dimension signal vector agent i sends to its neighbors, $\mathcal{S}_i$ denotes the variable-dimension collection of signals agent i receives from its neighbors through communication channels, the vector η_i is an estimator state containing information about the global performance of the system, and the vector y_i represents the output of the estimator (Figure 2). The formulation above expresses the estimator Q and R using continuous-time dynamics for simplicity, but sampled-data or hybrid dynamics could be substituted as appropriate.

Note that implicit in the formulation are three dynamic interaction networks: a network of physical interaction, represented by $\{\mathcal{X}_i^{\text{phys}}\}$; a network of sensing interaction, represented by $\{\mathcal{X}_i^{\text{sens}}\}$; and a network of communication interaction, implicit in $\{\mathcal{S}_i\}$. These networks may be changing over time.

[1]Not all of K, G, Q, R will necessarily use all parameters. Local state observers can be used to provide estimates $\hat{x}_i$ of x_i if required.

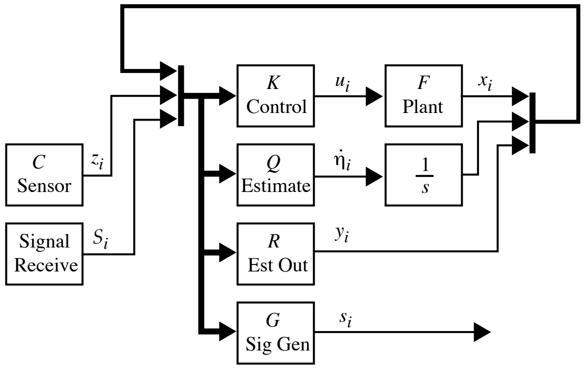


Fig. 2. Block diagram for agent i.

The goal of our work is to design a local controller K, a signal generator G, and a state estimator Q and R to minimize a cost functional J encoding the desired behavior of the system,

$$J = \phi(\mathcal{Y}(t_f), t_f) + \int_{t_0}^{t_f} \mathcal{L}(\mathcal{Y}(\tau), u(\tau), \tau)\, d\tau\,, \qquad (7)$$

where $u = [u_1^T \dots u_n^T]^T$ and $\mathcal{Y}$ represents the collection of all x_i, η_i, and y_i. Even when controllers cannot be designed to exactly minimize J, it can be used to measure the quality of a control design. This framework does not include sensing or communication costs, which may be important factors in certain applications.

A. The Need for Estimation

The need for the global state estimator Q and R is clear for the target localization problem—it is required to maintain estimates of the target location—but perhaps not so clear for the formation control problem. A natural question is whether purely reactive memoryless controllers can be used to solve the same problem. For example, one common approach to coordinated behavior relies solely on sensing interaction of the agents. In this framework, each agent might sense the positions and velocities of its neighbors and choose its motion based on this information. In this simplification, there is neither an estimator state η_i nor an estimator output y_i, and only a memoryless controller K is to be designed. Simple memoryless gradient control laws are effective when the spatial gradient of J can be shown to be *spatially distributed* over the sensing network—the gradient of J with respect to x_i is a function only of x_i and $\mathcal{X}_i^{\text{sens}}$ [6], [7]. For a particular cost J, the idea is to have the agents implement a sensing network over which the gradient of J is spatially distributed. Cortés et al. [6], [7] describe a number of possible network structures based on different notions of distance to neighbors. When the gradient of J is spatially distributed over none of these networks, we can instead use a cost J' that approximately encodes the desired behavior and whose gradient is spatially distributed over some network.

This approach requires agents to reliably implement a particular distance-based sensing network topology. Even when these networks are implemented reliably, the requirement of a spatially-distributed gradient of J is restrictive. We are interested in a broader class of tasks with fewer restrictions on the network topology. As an example, the following variance-control task, a simplified version of the formation control problem, is only spatially distributed over a fully-connected network. As we see later, a gradient-based controller can be effective even in the face of changing communication networks using simultaneous estimation and control.

Example 3 (Variance Control): Consider a network of n agents with $x_i \in \mathbb{R}$. Each agent has simple integrator dynamics

$$\dot{x}_i = u_i \in \mathbb{R}$$

and takes noise-free measurements of its own state as well as the states of its neighbors in the sensing network

$$z_i = \{x_i, \mathcal{X}_i^{\text{sens}}\}\,.$$

(Alternatively, each agent could simply measure the signed distance to its neighbors.) The desired behavior of the system is encoded by

$$J = \left[\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu(x))^2 - \sigma_d^2\right]^2,$$

where $\mu(x) = \frac{1}{n}\sum_i x_i$. The cost is minimized ($J = 0$) when the variance of the agents' states is σ_d^2. The system is clearly controllable to the $(n-1)$-dimensional goal set $\mathcal{M}$. However, the gradient of J is only spatially distributed over the fully-connected sensing network. We conjecture that there is no reactive memoryless controller K that stabilizes the goal exactly for all n and all possible connected sensing networks.[2] An interesting avenue for further research is to characterize the motion coordination problems that can and cannot be solved using purely reactive controllers.

B. Simultaneous Estimation and Control

To solve a broad class of decentralized sensing and control problems, including the problem above, we adopt the following four-step design methodology:

1) Choose a cost functional J encoding the desired performance of the system.
2) Design an initial local controller K^{initial} to achieve desired convergence properties. This controller may use global information on the performance of the system (and thus may be unimplementable), and it is often based on gradient descent.
3) Design a signal generator G and a global state estimator Q and R such that each agent can estimate the global information required by the controller K^{initial}. The estimator should provide correct estimates in steady state.
4) Construct the final control law K by replacing the global variables in K^{initial} by their local estimates, adding terms in the control law (if necessary) to preserve the desired convergence properties.

[2]Simple reactive controllers can be developed that *approximately* solve the task for arbitrary n and specific network structures, e.g., radius-limited sensing.

The design should be robust to changing network topologies and the addition and deletion of agents. It should also be scalable, meaning that the number of signals communicated by each agent $\dim(s_i)$ is independent of n.

The power of this approach rests on the capability of G, Q, and R to provide each agent with necessary estimates of the global performance of the system, subject to scalability constraints. In other words, each agent cannot simply pass around messages from all other agents with unique ID's attached to each message. Quantities that can be estimated in a scalable way include the minimum and maximum of state or sensed variables (each agent simply transmits the max or min value of its own sensor and any signal yet received) as well as averages of variables. An algorithm for decentralized consensus estimation of the average of static inputs was described in [34] and extended to changing inputs in [14], [44].

In this paper we focus on problems that can be solved using average consensus estimators. The class of functions that can be estimated using average consensus estimators is quite large, as we can first apply nonlinear functions to the variables, then pass them through the average estimator, and then apply nonlinear transformations to the result. For example, using a natural log transformation, average consensus estimators can be used to calculate the geometric mean of inputs, as opposed to the arithmetic mean. In the formation control example, formation moments are estimated using nonlinear transformations (powers) of agent position data.

To summarize, we typically choose the controller K in (3) based on the gradient of J with respect to x_i, the estimator Q and R in (5)–(6) to use dynamic average consensus estimation to estimate the quantities needed to implement K, and the signal generator G in (4) to broadcast agent estimates.

C. Related Work

The dynamic average consensus estimator integral to our work builds on the work of Olfati-Saber and Murray [34], who studied the convergence of the differential equation

$$\dot{x} = -Lx\,,$$

where x is a *decision variable* (e.g., an estimator state in our framework) and L is the *Laplacian* of a communication graph of agents (see Section III-C). They showed that each agent's decision variable x_i converges to a common value, and that this common value is the average of the initial values $x_i(0)$ if the communication graph is *balanced*, i.e., for each node in the graph, the in-degree and out-degree are equal. Average consensus is reached even with switching network topologies and communication delays bounded by a function of the largest eigenvalue of L. The rate of convergence of the consensus estimator as a function of the network topology and weights, as well as heuristic and analytic approaches to choosing network topologies and weights to optimize convergence rates, are studied in [30], [33], [47], [48].

We refer to this average consensus problem as *static*, as it incorporates only the initial data $x(0)$. To build an estimator capable of tracking the average of changing inputs, a high-pass *dynamic consensus estimator* was proposed in [44]. Dynamic consensus estimators with improved noise rejection and steady-state performance in the face of the addition and deletion of agents were introduced in [14]. These estimators are used in this paper to provide estimates of global performance. Dynamic consensus filters have also been utilized to construct decentralized Kalman filters [32].

In the present work we use average consensus estimators to explicitly estimate global properties of the system. These estimates are then used in the control law. Much past work has instead focused on using average consensus algorithms directly as the memoryless control law K. Typically agents sense the states of their neighbors and choose controls based on the average of these states. Such an approach can be applied to the problems of bringing agents to a common meeting point, or rendezvous [8], [24], and to various types of flocking and formation control [19], [25], [40]. Extensions of the basic idea to agents with second-order dynamics can be found in [22], [38]. Many of these algorithms are robust to a broad class of switching network topologies [19], [28].

While the simplicity of direct consensus control is appealing, the set of tasks to which it can be applied is limited. To design controllers for a broader class of tasks, Cortés et al. [6], [7] begin with an objective function J describing the desired behavior of the system. They call the gradient of the objective function *spatially distributed* over the graph if the gradient of the objective, as a function of agent i's position, can be calculated based only on the state of agent i and those of its neighbors. In cases where the gradient is spatially distributed, the gradient naturally suggests a control law. They show that some objective function gradients are spatially distributed over particular interaction graphs but not others, suggesting that the agents implement a sensing graph appropriate to the objective function.

The goal of our work is to use dynamic consensus estimators to expand the set of achievable collective behaviors beyond what can be achieved by reactive controllers. A similar idea was recently proposed by Scardovi and Sepulchre [42]. They illustrated the idea by demonstrating (almost) global phase synchronization of coupled oscillators in a connected, but not fully connected, network. Global synchronization is facilitated by basing each oscillator's dynamics on its estimate of the average of the initial phases of the oscillators. By choosing the inputs to the estimators to be constant, the estimator can be decoupled from the controller. The lack of feedback from the controller to the estimator allows convergence to be demonstrated without the need for small-gain or time-scale separation conditions. Porfiri et al. [36] proposed a decentralized estimate-and-control framework for formation tracking of a salinity contour, but the estimator leads to steady-state tracking error unless the communication network is fully connected.

The general framework for decentralized estimate-and-control is straightforward; the main results of this paper are in working through the details for the example of formation control.[3] The example of controlling moment statistics was previously studied with a centralized approach in [2]. This

[3]Portions of this work appeared in [16].

example was chosen in part because the gradient of the objective function is not spatially distributed over any network but the fully-connected one. In addition, formation control is one of the most-studied problems in decentralized control (see, e.g., [1], [10], [12], [20]–[23], [25], [26], [31], [38], [39], [45], [46]).

The second problem studied in this paper is to control the motion of each agent to maximize the expected sensory information on the location of a target, as a function of the range and bearing sensor model and the group's current Kalman-filter estimate of the location of the target. Our decentralized solution builds on the centralized control scheme in [5]. Zhou and Roumeliotis [49] use a centralized algorithm to solve the related problem of finding agent motions to minimize the trace of the covariance matrix of the agents' target location estimate.

The cost functionals considered in this paper have only terminal costs, allowing controllers based on simple gradient descent. For costs with an integral term, decentralized receding- or infinite-horizon control may be appropriate (see, for example, [29] and references therein).

III. Formation Control

A. Formulation

Suppose a swarm consists of a collection of n mobile agents having positions $p_1, \ldots, p_n \in \mathbb{R}^m$, which we write also as the combined vector $p = [p_1^T \ldots p_n^T]^T \in \mathbb{R}^{mn}$. Then we can represent the collection of all possible swarm configurations as the topological coproduct $\mathfrak{P} \triangleq \coprod_{n=1}^{\infty} \mathbb{R}^{mn}$. We describe the desired swarm configuration using a vector *goal function* $f : \mathfrak{P} \to \mathbb{R}^\ell$. The primary objective of each agent is to move itself to an equilibrium position so that the final swarm configuration p satisfies $f(p) = f^\star$, where $f^\star \in \mathrm{Im}(f)$ is a goal vector known to each agent.

The total number n of agents in the swarm is unknown to each agent, although the agents may have knowledge of an upper bound on n. Each agent measures its own position and velocity and controls its own acceleration, $\ddot{p}_i = u_i$. Furthermore, each agent can communicate with its neighbors; specifically, agents i and j can communicate with each other whenever $p_i \leftrightarrow p_j$, where $\leftrightarrow$ is a fixed symmetric relation on $\mathbb{R}^m$. For example, we may have $p_i \leftrightarrow p_j$ if and only if $|p_i - p_j| \leqslant r$, where r represents a fixed communication radius. Thus each configuration $p \in \mathfrak{P}$ defines the graph of an underlying communication network, and we let $\mathfrak{C} \subset \mathfrak{P}$ denote the set of all such configurations for which this graph is connected. As the agents move with time, the topology of this network can change, but we will perform our stability analysis below under the assumption that $p(t) \in \mathfrak{C}$, namely, that the network remains connected in forward time. For this reason we will assume $f^\star \in f(\mathfrak{C})$.

Our approach is based on following estimates of the gradient ∇J of the potential function $J : \mathfrak{P} \to \mathbb{R}$ given by

$$J(p) = \big[f(p) - f^\star\big]^T \Gamma \big[f(p) - f^\star\big] , \tag{8}$$

where $\Gamma \in \mathbb{R}^{\ell \times \ell}$ is a suitably chosen symmetric positive-definite global gain matrix. We let the set

$$\mathrm{Crit}(J) \triangleq \big\{p \in \mathfrak{P} : \nabla J(p) = 0\big\} \tag{9}$$

denote the set of critical points of J, and we classify such points as "good" critical points where $f(p) = f^\star$ (these are the global minima of J) and "bad" critical points where $f(p) \neq f^\star$. We want the swarm to avoid getting stuck at bad critical points. Unfortunately, even if a bad critical point of a C^∞ potential J is a strict local maximum of J, it can still be a stable equilibrium of the associated gradient flow $\dot{p} = -\nabla J(p)$. For example, suppose $J : \mathbb{R} \to \mathbb{R}$ is the C^∞ function

$$J(p) = \left[1 - \int_0^p x \, \exp\!\big(-\frac{1}{x^2}\big) \, \cos^2\!\big(\frac{1}{x^2}\big) \, dx\right]^2 . \tag{10}$$

This function $J(p)$ has a strict local maximum at $p = 0$, but this point is not isolated in $\mathrm{Crit}(J)$ and is in fact a stable equilibrium of the gradient flow. To rule out such pathological behavior, we will assume that J is locally constant on $\mathrm{Crit}(J)$.[4] This will indeed be the case for potentials of the form (8) when the goal function f is *subanalytic* (see the Appendix) or in particular when f is a polynomial function.

Our algorithms will guarantee that the swarm trajectories always converge to equilibrium sets.[5] For this reason we want all positive limit sets containing bad critical points of J to be "unstable" in the following sense:

Definition 1: Let $\pi(t, x)$ be a continuous stationary flow on a topological space $\mathcal{X}$. A closed, π-invariant set $L \subset \mathcal{X}$ is *strongly unsteady* (respectively, *weakly unsteady*) when there exists an open set $\mathcal{O} \subset \mathcal{X}$ with $L \subset \mathcal{O}$ such that for any open set $\mathcal{U} \subset \mathcal{X}$ with $L \cap \mathcal{U} \neq \varnothing$ (respectively, with $L \subset \mathcal{U}$), there exists an initial state $x_0 \in \mathcal{U}$ and a time $T \geqslant 0$ such that $\pi(t, x_0) \in \mathcal{X} \setminus \mathcal{O}$ for all $t \geqslant T$.

Clearly all strongly unsteady invariant sets are weakly unsteady, and the two notions coincide for equilibria. A weakly unsteady invariant set is both unstable (in the sense of Lyapunov) and unattractive, but the converse is not always true (for example, one can have an unstable, unattractive equilibrium which is not unsteady). If all positive limits sets containing bad critical points are strongly unsteady, then whenever a swarm trajectory approaches such a limit set, a small perturbation will cause it to leave a neighborhood of this set forever.

B. Moment Statistics

We focus on goal functions f of the form

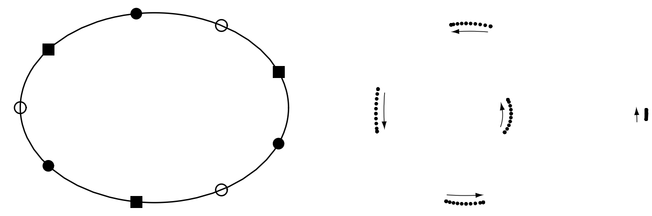
$$f(p) = \frac{1}{n(p)} \sum_{i=1}^{n(p)} \phi(p_i) , \tag{11}$$

where $\phi : \mathbb{R}^m \to \mathbb{R}^\ell$ is a given *moment-generating function* and $n(p)$ is the unique integer n such that $p \in \mathbb{R}^{mn}$. For example, for $m = 3$ and $p_i = [p_{ix} \; p_{iy} \; p_{iz}]^T$, this function ϕ might be a list of ℓ monomials of the form

$$\phi(p_i) = p_{ix}^a \, p_{iy}^b \, p_{iz}^c , \tag{12}$$

[4]We say that a function f on a topological space $\mathcal{X}$ is *locally constant on a set* $S \subset \mathcal{X}$ when every $x \in S$ has an open neighborhood $N \subset \mathcal{X}$ such that f is constant on $N \cap S$.

[5]Note that convergence to an equilibrium set does not guarantee convergence to a single equilibrium, even in gradient systems.



(a) (b)

Fig. 3. (a) For three robots in the plane ($mn = 6$), first- and second-order moment constraints ($\ell = 5$) restrict the swarm formation to a one-dimensional space. Shown are three example formations, where the robots in the same formation share the same symbol. The robots are confined to an ellipse determined by the constraints. (b) For five robots in the plane ($mn = 10$), first-, second-, and third-order moment constraints ($\ell = 9$) restrict the swarm formation to a one-dimensional space. The robots are shown moving along the constraint-preserving set.

where a, b, and c are nonnegative integers. In this case the goal function (11) is a list of ℓ *moments* of the form

$$M_{abc} = \frac{1}{n}\sum_{i=1}^{n} p_{ix}^a\, p_{iy}^b\, p_{iz}^c\,, \tag{13}$$

where the sum $a+b+c$ is called the *order* of the moment. Given a particular swarm formation, a sufficient number of moments is guaranteed to distinguish it from any other formation. In other words, moments can provide an exact formation description. We are interested, however, in the case where a small number of low-order moments is used to specify a family of formations. If ℓ moment constraints are specified on n robots in an m-dimensional space, in general there is an $(mn-\ell)$-dimensional algebraic set of swarm configurations that satisfy the constraints. Examples of one-dimensional families of swarm configurations are given in Figure 3. The structure and topology of such formation families can be studied using tools from real algebraic geometry [4].

Our primary example in this paper involves formations defined by first- and second-order moments. The m first-order moments specify the center of mass of the swarm. From the $m(m+1)/2$ second-order moments we can derive $m(m-1)/2$ variables describing the orientation of orthogonal principal axes of inertia of the swarm and m shape variables summarizing the elongation of the swarm along the principal axes. Our abstraction of the swarm formation, then, is given by the $m(m+1)/2$ group variables describing the position and orientation of the principal axis frame in $SE(m)$ and the m shape variables describing the elongation of the swarm along these axes [2].

To write the moment-generating function ϕ for first- and second-order moments, let $\operatorname{triu} : \mathbb{R}^{m\times m} \to \mathbb{R}^{m(m+1)/2}$ denote the linear map which stacks the main and upper diagonals of a matrix into a vector, so that for $m = 3$,

$$\operatorname{triu}\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} = \begin{bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ \alpha_{12} \\ \alpha_{23} \\ \alpha_{13} \end{bmatrix}. \tag{14}$$

Then ϕ for first- and second-order moments can be written as

$$\phi(p_i) = \begin{bmatrix} p_i \\ \operatorname{triu}(p_i p_i^T) \end{bmatrix} \in \mathbb{R}^{\ell}\,, \tag{15}$$

where $\ell = m(m+3)/2$.

Given a closed set of swarm configurations $\mathcal{D} \subset \mathfrak{P}$ and a goal vector $f^\star \in f(\mathcal{D})$, we let $\mathcal{G}(f^\star, \mathcal{D})$ denote the cone of all symmetric positive definite matrices Γ such that no bad critical points of J in $\mathcal{D}$ are local minima of J. To reduce the risk of the swarm getting stuck at a bad critical point of J, we would ideally choose a weighting matrix Γ belonging to $\mathcal{G}(f^\star, \mathcal{D})$ for a large set $\mathcal{D}$ (i.e., one containing all likely swarm configurations). However, it may be difficult to find such matrices Γ for general goal functions f. Nevertheless, for the case of formations defined by first- and second-order moments with ϕ as in (15), we can always compute members of $\mathcal{G}(f^\star, \mathcal{D})$ when $\mathcal{D}$ contains all possible configurations of at least $m+1$ agents:

Theorem 2: Let ϕ be as in (15), let $\mathcal{D} = \bigcup_{n=m+1}^{\infty} \mathbb{R}^{mn}$, and let $f^\star \in f(\mathcal{D})$. Then there exists a symmetric matrix $\Gamma > 0$ such that for every bad critical point $p \in \mathcal{D}$ of J, the Hessian matrix $\mathcal{H}J(p)$ has at least one strictly negative eigenvalue. In particular, $\Gamma \in \mathcal{G}(f^\star, \mathcal{D})$.

The proof of this theorem, which is constructive, is provided in the Appendix.

C. Nonlinear Gradient Control with High-Pass Estimators

In the notation of Section II, the physical state of agent i is $x_i = [p_i^T\ \dot{p}_i^T]^T$, with dynamics

$$\dot{x}_i = F(x_i, u_i) = \begin{bmatrix} \dot{p}_i \\ u_i \end{bmatrix} \tag{16}$$

and noise-free measurements

$$z_i = C(x_i, \mathcal{X}_i^{\text{sens}}) = \begin{bmatrix} x_i \\ f^\star \end{bmatrix}. \tag{17}$$

We have already completed the first step in the design methodology of Section II-B by choosing the cost J in (8) with $f(p)$ in (11). According to the second step, we choose an initial (unimplementable) local controller $u_i = K^{\text{initial}}$ based on the gradient of this cost, with an additional damping term:

$$\begin{aligned} u_i &= K^{\text{initial}}(p, \dot{p}_i, f^\star) \\ &= -B\dot{p}_i - \big[\mathcal{J}\phi(p_i)\big]^T \Gamma \big[f(p) - f^\star\big]\,, \end{aligned} \tag{18}$$

where $B \in \mathbb{R}^{m\times m}$ is a damping matrix and $\mathcal{J}\phi(\cdot)$ denotes the $\ell \times m$ Jacobian matrix of ϕ. Here $f(p)$ represents global information unavailable to each agent, so according to the third step in our methodology, we design signal generator G and a global state estimator Q and R to provide local estimates y_i of the global variable $f(p)$. In this section we consider the following signal generator and estimator:

$$s_i = G(x_i, z_i, \eta_i, y_i, \mathcal{S}_i) = \begin{bmatrix} p_i \\ y_i \end{bmatrix} \tag{19}$$

$$\begin{aligned} \dot{\eta}_i &= Q(x_i, z_i, \eta_i, y_i, \mathcal{S}_i) \\ &= -\gamma\eta_i - \sum_{j\neq i} a(p_i, p_j)\big[y_i - y_j\big] \end{aligned} \tag{20}$$

$$y_i = R(x_i, z_i, \eta_i, \mathcal{S}_i) = \eta_i + \phi(p_i)\,. \tag{21}$$

Here $y_i(t) \in \mathbb{R}^\ell$ is the agent's current estimate of $f(p)$ and $\eta_i(t) \in \mathbb{R}^\ell$ is the internal estimator state. To implement this estimation algorithm, each agent i must continually transmit its current values of p_i and y_i to its neighbors, as indicated by the signal generator (19). In the estimator dynamics (20), $\gamma \geqslant 0$ is an estimator "forgetting factor" and $a : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is a C^1 symmetric function[6] such that $\operatorname{supp}(a) \subset \operatorname{Graph}(\leftrightarrow)$ (so that $a(p_i, p_j) \neq 0$ only if agents i and j can communicate with each other). We call the estimator (20)–(21) a high-pass estimator because if the estimator gains $a(p_i, p_j)$ were constant, then the resulting LTI system taking the inputs $\phi(p_i)$ to the outputs y_i would be a high-pass filter with unity high-frequency gain. This estimator is introduced in [14] and is based on the one in [44].

The fourth and final step in our design is to construct the actual local control law K by replacing $f(p)$ in (18) with y_i and adding a stabilizing nonlinear damping term:

$$\begin{aligned} u_i &= K(x_i, z_i, \eta_i, y_i, \mathcal{S}_i) \\ &= -B\dot{p}_i - \big[\partial\phi(p_i)\big]^T \Gamma \big[y_i - f^\star\big] \\ &\qquad - \big[\partial\phi(p_i)\big]^T \Lambda \big[\partial\phi(p_i)\big]\dot{p}_i \,, \end{aligned} \tag{22}$$

where $\Lambda \in \mathbb{R}^{\ell\times\ell}$ is a damping gain matrix. The utility of the extra nonlinear damping term is apparent in the proof (in the Appendix) of Theorem 3, stated below. We now proceed to investigate the behavior of this scheme (19)–(22).

Suppose the number n of agents in the swarm is fixed. We let $\mathbf{1}_n \in \mathbb{R}^n$ denote the vector of n ones (or simply $\mathbf{1}$ when n is clear from context), and we let $\operatorname{Orth}(\mathbf{1})$ denote the collection of $n \times (n-1)$ matrices S such that $S^T S = I$ and $S^T \mathbf{1} = 0$ (namely, the columns of S form an orthonormal basis for $\operatorname{span}\{\mathbf{1}\}^\perp$). Then by orthogonal decomposition,

$$I = SS^T + \frac{\mathbf{1}\mathbf{1}^T}{n} \tag{23}$$

and thus $ASS^TA^T \leqslant AA^T$ for any n-column real matrix A (in particular we have $|AS|_F \leqslant |A|_F$ where $|\cdot|_F$ denotes the Frobenius norm). We define the *Laplacian* $L(p) \in \mathbb{R}^{n\times n}$ to be the symmetric matrix whose off-diagonal elements in row i, column j are equal to $-a(p_i, p_j)$ and whose diagonal elements are the negatives of the sums of the off-diagonal elements in the same row (so that $L(p)\mathbf{1} \equiv 0$). Moreover, fixing some $S \in \operatorname{Orth}(\mathbf{1})$, we define the *reduced Laplacian* $L^\star(p)$ to be the $(n-1)\times(n-1)$ symmetric matrix

$$L^\star(p) = S^T L(p) S \,, \tag{24}$$

and we note from (23) that $SL^\star(p) = L(p)S$. Furthermore, for a connected configuration $p \in \mathfrak{C}$ and for positive estimator weights $a(\cdot,\cdot)$ on the connected arcs, the smallest eigenvalue of the reduced Laplacian $L^\star(p)$ (called the *algebraic connectivity* of the underlying graph) will be strictly positive [11], [13]. The first of our two primary assumptions we use to prove our convergence results is that this eigenvalue is bounded away from zero, i.e., that

$$L^\star(p) \geqslant \varepsilon I \tag{25}$$

[6]If $\mathcal{X}$ and $\mathcal{Y}$ are nonempty sets, we say that a function $\psi : \mathcal{X}\times\mathcal{X} \to \mathcal{Y}$ is *symmetric* when $\psi(a,b) = \psi(b,a)$ for all $a, b \in \mathcal{X}$.

along trajectories in forward time for some constant $\varepsilon > 0$. In particular, (25) implies that $p(t) \in \mathfrak{C}$ for all $t \geqslant t_0$. The second of our two primary assumptions takes the form of the small-gain condition

$$\lambda_{\max}(\Gamma) < \frac{\varepsilon}{2}\,\lambda_{\min}\big(\Lambda + \Lambda^T\big)\,, \tag{26}$$

where $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote the maximum and minimum eigenvalues (respectively). To better understand these conditions (25) and (26), suppose the estimator gain function $a(\cdot,\cdot)$ in (20) is simply

$$a(p_i, p_j) = \begin{cases} a_0 & \text{when } p_i \leftrightarrow p_j \\ 0 & \text{otherwise,} \end{cases} \tag{27}$$

where $a_0 > 0$ is a scalar constant estimator gain (technically, $a(\cdot,\cdot)$ would be a C^1 approximation to this simple choice). Then for a connected network, the value of ε is bounded from below by $2a_0 - 2a_0\cos(\pi/n)$, its value for the worst-case configuration of a linear chain of agents [11]. Thus if we know an upper bound $n_{\max}$ on the total number n of agents in the swarm, then we can choose our gains to satisfy

$$\lambda_{\max}(\Gamma) < a_0\Big[1 - \cos\big(\frac{\pi}{n_{\max}}\big)\Big]\,\lambda_{\min}\big(\Lambda + \Lambda^T\big)\,. \tag{28}$$

In this case we will always satisfy (25) and (26) provided the network remains connected. The inequality (28) involves the gradient-descent control gain Γ, the estimator gain a_0, and the nonlinear damping gain Λ. Because of the possibility of noise and delay in the communication channels, we would not choose the estimator gain a_0 to be too large, which means we would satisfy (28) by moving slowly enough, either by choosing a small control gain Γ or a large damping gain Λ.

The following theorem states our results for the case that $\gamma = 0$ and that each state η_i has the initial value $\eta_i(t_0) = 0$; the more general cases will be discussed below.

Theorem 3: Suppose ϕ is C^2 and proper, fix $f^\star \in f(\mathfrak{C})$, suppose $B + B^T > 0$, suppose $\eta_i(t_0) = 0$ for each i, and suppose the weighting $a(\cdot,\cdot)$ is C^1 and symmetric. Suppose n is fixed, suppose (25) and (26) hold for some $\varepsilon > 0$, and fix $\gamma = 0$. Then each trajectory of the swarm system (19)–(22) is bounded in forward time, and its positive limit set L^+ consists of equilibria. If in addition ϕ is subanalytic and there exists a closed set $\mathcal{D} \subset \mathfrak{P}$ such that $\Gamma \in \mathcal{G}(f^\star, \mathcal{D})$ and $p(t) \in \mathcal{D}$ for all $t \geqslant t_0$, then every positive limit set L^+ containing a bad equilibrium (i.e., an equilibrium corresponding to a bad critical point of J) is strongly unsteady.

In particular, if we choose ϕ as in (15) to include all first- and second-order moments, if we assume $n \geqslant m+1$, and if we choose Γ and $\mathcal{D}$ according to Theorem 2, then clearly ϕ is C^2, proper, and subanalytic, $\Gamma \in \mathcal{G}(f^\star, \mathcal{D})$, and $p(t) \in \mathcal{D}$ for all $t \geqslant t_0$. In this case the swarm will generically converge to the set of configurations satisfying the desired moment statistics, leaving any bad configuration after a slight perturbation.

As is evident in the proof of this theorem in the Appendix, the dynamics of the estimator (20)–(21) include a subsystem of the form $\dot{\chi} = -\gamma\chi$ which is uncontrollable from the inputs $\phi(p_i)$ but observable through the estimation errors $e_i = f(p) - y_i$. If $\gamma = 0$ and if the states $\eta_i(t_0)$ are not initialized to zero, then the constants χ will generate persistent nonzero constant offsets in the error variables e_i. These steady-state estimation errors will cause the swarm to converge to a formation with the wrong statistics. To avoid such errors, one would have to somehow globally simultaneously reinitialize these states to zero whenever agents leave the swarm (e.g., due to failure) or new agents join the swarm. Furthermore, if $\gamma = 0$ then any additive noise in the communication channels will pass through pure integrators $\dot{\chi} = \text{noise}$, resulting in random drift in the estimation errors. To alleviate these problems one could choose $\gamma > 0$; in this case any incorrect initialization of the states $\eta_i(t_0)$ will be asymptotically forgotten, and communication noise will not cause random drift. However, the estimator (20)–(21) exhibits steady-state error under constant inputs, an error whose size is proportional to $\gamma/(\gamma + \varepsilon)$ (and hence nonzero for $\gamma > 0$) [14]. Nevertheless, as we will illustrate in Section III-E, a small error due to a small positive γ may be preferable to errors caused by incorrect initializations. In the next subsection we introduce a more complex estimator which achieves robustness to initialization errors and adding or subtracting agents from the network but does so without introducing any steady-state error.

The conclusion of Theorem 3 (and likewise of Theorem 4 below) remains valid if the damping matrix B is a C^1 function of the states $x_1, \ldots, x_n$ and $\eta_1, \ldots, \eta_n$, provided $B(\cdot)+B^T(\cdot) > 0$ holds globally (however, keep in mind that B can only depend on local variables, i.e., variables available to each agent via sensing or communication). Hence we can view this damping matrix B as an additional source of control, and we might design it to help maintain network connectivity or to help avoid collisions between agents. This extension is a topic of future research.

D. Nonlinear Gradient Control with PI Estimators

In this section we assume that there exists a proper metric d on $\mathbb{R}^m$ such that the quantity

$$\mathfrak{d}(n) \triangleq \sup_{p\in\mathfrak{C}\cap\mathbb{R}^{mn}} \max_{1\leqslant i,j\leqslant n} d(p_i, p_j)\,, \tag{29}$$

which is the maximum diameter of a connected swarm of n agents, is finite for every n. For the case in which $p_i \leftrightarrow p_j$ if and only if $|p_i - p_j| \leqslant r$, where $r > 0$ is a fixed communication radius, we can take d to be the usual Euclidean metric on $\mathbb{R}^m$. It follows from (29) that there exists a class-$\mathcal{K}$ function $\mathfrak{a}$ and a C^1 function $\zeta : \mathbb{R}^m \to \mathbb{R}$ such that

$$|\phi(p_i) - \phi(p_j)|^2 \leqslant \mathfrak{a}\big(\mathfrak{d}(n(p))\big) \cdot \zeta(p_i) \tag{30}$$

for every $p \in \mathfrak{C}$ and every $i, j \in \{1, \ldots, n(p)\}$ [15, Corollary A.15].

The agent dynamics, measurements, and initial local controller are as before in (16), (17), and (18), but now we use a new signal generator and estimator as follows:

$$s_i = \begin{bmatrix} p_i \\ \eta_i \end{bmatrix}, \qquad \text{where } \eta_i = \begin{bmatrix} v_i \\ w_i \end{bmatrix} \tag{31}$$

$$\begin{aligned} \dot{v}_i = &-\gamma v_i - \sum_{j\neq i} a(p_i,p_j)\big[v_i - v_j\big] \\ &+ \sum_{j\neq i} b(p_i,p_j)\big[w_i - w_j\big] + \gamma\phi(p_i) \end{aligned} \tag{32}$$

$$\dot{w}_i = -\sum_{j\neq i} b(p_i,p_j)\big[v_i - v_j\big] \tag{33}$$

$$y_i = v_i\,. \tag{34}$$

Here $\gamma > 0$ is a global forgetting factor which controls the rate of replacing old information with new (with $\gamma = 0$ no longer allowed as it now scales the input to the estimator), and $a, b : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ are bounded C^1 symmetric functions such that $\operatorname{supp}(a) \cup \operatorname{supp}(b) \subset \operatorname{Graph}(\leftrightarrow)$. We also assume that b has bounded first-order partial derivatives. We call the estimator (32)–(34) a PI estimator because of the presence of the integral w_i of the weighted differences between local estimates (33). Unlike the high-pass estimator (20)–(21), this PI estimator has no direct feedthrough from its input $\phi(p_i)$ to its output y_i, and thus it may provide better filtering of high-frequency noise.

The actual control law for use with this new PI estimator is similar to the one in (22) but includes an additional nonlinear damping term:

$$\begin{aligned} u_i = &-B\dot{p}_i - \big[\partial\phi(p_i)\big]^T\Gamma\big[y_i - f^\star\big] \\ &- \big[\partial\phi(p_i)\big]^T\Lambda\big[\partial\phi(p_i)\big]\dot{p}_i - c\zeta(p_i)\dot{p}_i\,, \end{aligned} \tag{35}$$

where B and Λ are damping gain matrices as before and $c > 0$ is a new scalar nonlinear damping gain. We now proceed to investigate the behavior of this scheme (31)–(35).

Again, suppose n is fixed. We define the *proportional Laplacian* $L_P(p) \in \mathbb{R}^{n\times n}$ to be the symmetric matrix whose off-diagonal elements in row i, column j are equal to $-a(p_i, p_j)$ and whose diagonal elements are such that $L_P(p)\mathbf{1} \equiv 0$. We define the *integral Laplacian* $L_I(p) \in \mathbb{R}^{n\times n}$ in the same way but using $b(\cdot,\cdot)$ instead of $a(\cdot,\cdot)$. Again fixing $S \in \operatorname{Orth}(\mathbf{1})$, we define the corresponding reduced Laplacians $L_P^\star(p) = S^T L_P(p) S$ and $L_I^\star(p) = S^T L_I(p) S$. Our first primary assumption we use to prove our convergence results is that there exist constants $\rho > -\gamma$ and $\varepsilon > 0$ such that

$$\rho I \leqslant L_P^\star(p) \leqslant \bar{\rho} I \tag{36}$$

$$\varepsilon I \leqslant L_I^\star(p) \leqslant \bar{\varepsilon} I \tag{37}$$

along trajectories in forward time (again implying a connected network $p(t) \in \mathfrak{C}$). Here the constants $\bar{\rho}, \bar{\varepsilon} > 0$ represent upper bounds on the reduced Laplacians which exist because the functions a and b are bounded. Notice that ρ need not be a positive number; in particular, the choice $a(\cdot,\cdot) \equiv 0$ results in $L_P^\star(\cdot) \equiv 0$ which satisfies (36) with $\rho = 0$. Such a choice simplifies the estimator without changing our convergence results, but it might adversely impact performance.

Our second primary assumption takes the form of the small-gain condition

$$\lambda_{\max}(\Gamma) < \delta_1 \lambda_{\min}\big(\Lambda + \Lambda^T\big) \leqslant \delta_2 c\,, \tag{38}$$

where $\delta_1, \delta_2 > 0$ are scalar constants depending on n, ρ, $\bar{\rho}$, ε, $\bar{\varepsilon}$, γ, and the bounds on the partial derivatives of b (the exact dependencies are provided in the proof of Theorem 4 in the Appendix). As before, if an upper bound on n is known, then we can compute gains Γ, Λ, and c which satisfy (38).

Theorem 4: Suppose ϕ is C^2 and proper, fix $f^\star \in f(\mathfrak{C})$, suppose $B + B^T > 0$, and suppose $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are C^1, bounded, symmetric, and such that b has bounded first-order partial derivatives. Suppose n is fixed, suppose (36) and (37) hold for some $\rho > -\gamma$ and $\varepsilon > 0$ (with $\gamma > 0$), and suppose the inequalities (38) are satisfied. Then each trajectory of the swarm system (31)–(35) is bounded in forward time, and its positive limit set L^+ consists of equilibria. If in addition ϕ is subanalytic and there exists a closed set $\mathcal{D} \subset \mathfrak{P}$ such that $\Gamma \in \mathcal{G}(f^\star, \mathcal{D})$ and $p(t) \in \mathcal{D}$ for all $t \geqslant t_0$, then every positive limit set L^+ containing a bad equilibrium is strongly unsteady.

Like the high-pass estimator (20)–(21) with $\gamma = 0$, the PI estimator (32)–(34) (with $\gamma > 0$) includes a subsystem of the form $\dot{\chi} = 0$ which is uncontrollable from the inputs $\phi(p_i)$ (see the proof in the Appendix). Thus, as before, χ might be nonzero due to inconsistent initializations and might drift due to communication noise. However, unlike the high-pass case, these states χ are not observable through the estimation errors $e_i = f(p) - y_i$, which means their behavior will not affect the swarm dynamics.

E. Simulation Results

We simulated the algorithms in Sections III-C and III-D for a swarm of $n = 7$ planar robots ($m = 2$), ϕ as in (15), and $f^\star = [0 \quad 0 \quad 50 \quad 0 \quad 50]^T$. The controller gain matrix was $\Gamma = \mathrm{diag}(80, 80, 8, 8, 8)$. The estimator gain functions where chosen according to an equal weighting scheme with a communication radius of 15: $a(p_i, p_j) = a_0$ and $b(p_i, p_j) = b_0$ when $|p_i - p_j| \leq 15$ and $a(p_i, p_j) = b(p_i, p_j) = 0$ otherwise (the fact that these gain functions are discontinuous had little effect on the simulations). Also, we set the nonlinear damping gains Λ and c in (22) and (35) to zero as the constant B provided adequate damping over a bounded region.

We first simulated the high-pass scheme of Section III-C with damping $B = 40I$, estimator gain $a_0 = 20$, and no forgetting factor ($\gamma = 0$). Figure 4 shows the results of the inertial moments M_{10} = CMx (the first component of f) and M_{02} = Ixx (the fifth component of f). The first 25 seconds show the convergence of the formation statistics to their desired values with no steady-state error. At time $t = 25$, one of the agents fails and leaves the swarm, resulting in a permanent nonzero steady-state error after that point. Actually, the remaining agents do not move at all from their equilibria after time $t = 25$, demonstrating that the high-pass estimator with no forgetting factor does not recover from initialization errors. If we include a nonzero forgetting factor of $\gamma = 0.3$, then we do recover from the loss of the agent (Figure 5), but we now incur a small nonzero steady-state error both before and after the loss.

We next simulated the PI scheme of Section III-D with increased damping $B = 100I$, estimator gains $a_0 = 20$ and $b_0 = 0.2$, and $\gamma = 6$. Figures 6 and 7 show that the PI algorithm can also recover from the loss of an agent (again at time $t = 25$) but now with zero steady-state error.

F. Kinematic Agents

If we control the agent velocities rather than their accelerations, we can still obtain appropriate control laws for use with either the high-pass or the PI estimators. For example, in the

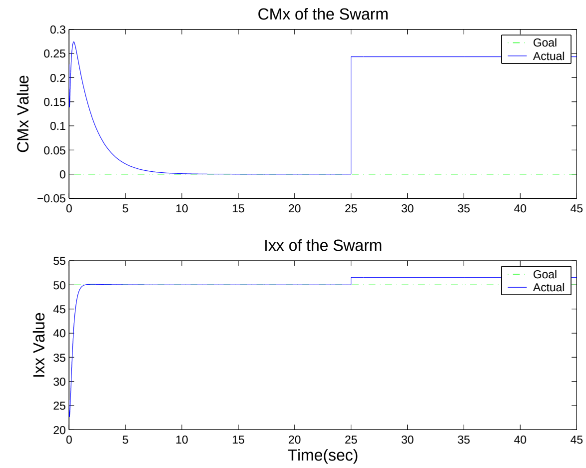


Fig. 4. The high-pass algorithm with no forgetting factor ($\gamma = 0$).

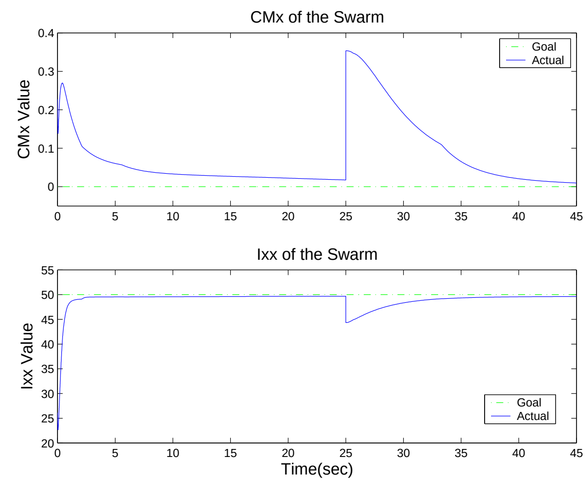


Fig. 5. The high-pass algorithm with forgetting factor $\gamma = 0.3$.

case of the high-pass estimator with kinematic agents $\dot{p}_i = u_i$, we can use the local controller

$$u_i = -\Big[B + \big[\partial\phi(p_i)\big]^T \Lambda \big[\partial\phi(p_i)\big]\Big]^{-1} \big[\partial\phi(p_i)\big]^T \Gamma \big[y_i - f^\star\big] .$$

Motivated by a singular perturbation analysis, we obtained this controller from the one in (22) by setting the right-hand side of (22) to zero (i.e., setting $\ddot{p}_i = 0$) and solving for $\dot{p}_i$. The convergence properties remain the same as those described in Theorem 3 above; in fact we obtain the same dissipation equality (107) we derived in the proof of this theorem in the Appendix, albeit with a storage function $V(p) = nJ(p)$ which does not include a kinetic energy term.

IV. Cooperative Target Localization

In this application a group of mobile sensors cooperatively tracks the location of a target, and the goal of each agent is to move in such a way as to minimize the uncertainty in the fused sensor reading. We obtain the local controllers following the general design methodology outlined in Section II-B. In contrast to the formation control problem we examined in Section III, here we assume that each agent knows the total number n of agents in the swarm.

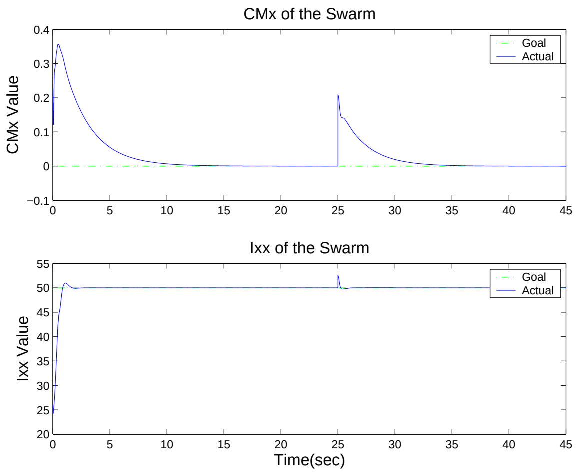


Fig. 6. The PI algorithm.

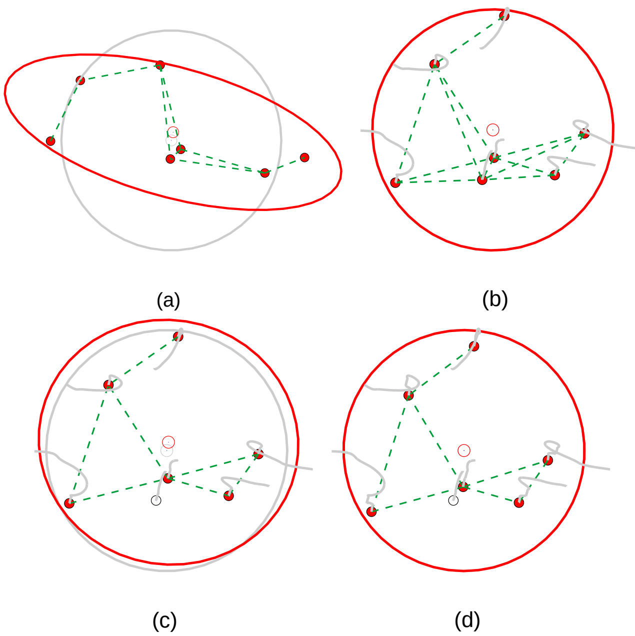


Fig. 7. Snapshots of the robots' movement under the PI scheme: (a) $t = 0$ initial condition, (b) $t = 25$ robots satisfy the goal, (c) $t = 25$ one robot dies, (d) $t = 45$ robots move to re-satisfy the goal.

A. Formulation

Following [5], we consider n sensors and one target moving in the plane, having positions $p_1, \cdots, p_n \in \mathbb{R}^2$ and $x_t \in \mathbb{R}^2$, respectively. The observation made by the i^{th} sensor is given by the linear measurement model

$$z_i = x_t + v_i\,, \qquad i = 1, \ldots, n\,, \tag{39}$$

and the measurement noise v_i is a continuous-time Gaussian noise with zero mean. In keeping with the standard range-finding sensor model [37], its covariance matrix R_i assumes a diagonal structure in the sensor's local range/bearing frame:

$$R_i = \begin{bmatrix} (\sigma^i_{\text{range}})^2 & 0 \\ 0 & (\sigma^i_{\text{bearing}})^2 \end{bmatrix}. \tag{40}$$

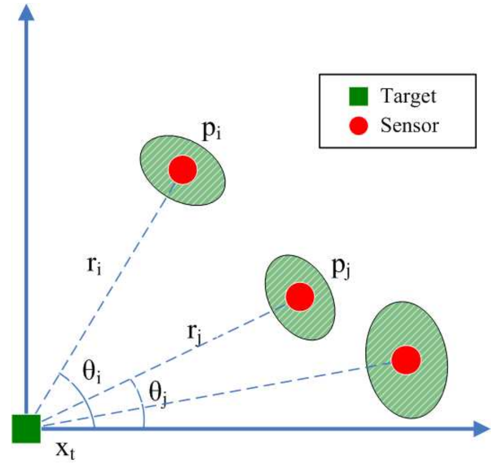


Fig. 8. Schematic of the sensor model.

The range measurement noise variance $(\sigma^i_{\text{range}})^2$ is commonly represented by a function $f_r(r_i)$ of the distance r_i from the target to sensor i. The bearing noise variance $(\sigma^i_{\text{bearing}})^2$ also depends on the range and can be modeled as $f_b(r_i)$. We use the following simple yet representative forms of these functions:

$$(\sigma^i_{\text{range}})^2 = f_r(r_i) = a_2(r_i - a_1)^2 + a_0$$
$$(\sigma^i_{\text{bearing}})^2 = f_b(r_i) = \alpha f_r(r_i)\,,$$

where a_0, a_1, a_2, α are model parameters. This measurement uncertainty model assumes the existence of a "sweet spot" location $r_i = a_1$ at which the noise is at its minimum value.

We will consider two different ways of fusing the local target position measurements z_i and error covariances R_i to obtain a global target position estimate $\hat{x}_{\text{global}}$ and global error covariance P_{global}. The first method, described in Section IV-B and based on the work in [5], uses only current measurements to obtain $\hat{x}_{\text{global}}$ and P_{global}. The second method, described in Section IV-C, defines $\hat{x}_{\text{global}}$ and P_{global} by means of a Kalman filter. In either case, the matrix P_{global} depends on the sensor and target locations, which means the sensors can move to reduce the cost

$$J = \det(P_{\text{global}})\,. \tag{41}$$

This cost was used in [5] as a measure of global uncertainty; other suitable measures discussed in [27] include the trace of P_{global} [49]. For simplicity, we assume all agents are kinematic and fully actuated so that $\dot{p}_i = u_i$, and we consider the initial (unimplementable) local gradient controller

$$u_i = K^{\text{initial}}(\cdot) = -\Gamma T_i^T \begin{bmatrix} \dfrac{\partial J}{\partial r_i} \\ \dfrac{1}{r_i}\dfrac{\partial J}{\partial \theta_i} \end{bmatrix}, \tag{42}$$

where $\Gamma > 0$ is a gain matrix, $\theta_i = \angle(p_i - x_t)$ is the angle from the target to sensor i, and T_i is the rotation matrix

$$T_i = \begin{bmatrix} \cos(\theta_i) & \sin(\theta_i) \\ -\sin(\theta_i) & \cos(\theta_i) \end{bmatrix} \tag{43}$$

which transforms R_i, the covariance matrix in the local frame, to $T_i R_i T_i^T$, the covariance matrix in the global Cartesian frame. Defining

$$P_i^r \triangleq \frac{\partial P_{\text{global}}}{\partial r_i}\,, \qquad P_i^\theta \triangleq \frac{\partial P_{\text{global}}}{\partial \theta_i}\,, \tag{44}$$

we can calculate the gradients in (42) as

$$\frac{\partial J}{\partial r_i} = J \cdot \operatorname{Tr}\big[P_{\text{global}}^{-1} P_i^r\big] \tag{45}$$

$$\frac{\partial J}{\partial \theta_i} = J \cdot \operatorname{Tr}\big[P_{\text{global}}^{-1} P_i^\theta\big]\,. \tag{46}$$

We will obtain the implementable, decentralized local controller $u_i = K(\cdot)$ from (42) by replacing any unavailable global quantities with local estimates.

B. One-Time Measurement Approach

An instantaneous fusion of current sensor readings leads to the following relations [5], [37]:

$$P_{\text{global}}^{-1}\hat{x}_{\text{global}} = \sum_{i=1}^{n}(T_i R_i T_i^T)^{-1} z_i = \sum_{i=1}^{n} T_i R_i^{-1} T_i^T z_i \tag{47}$$

$$P_{\text{global}}^{-1} = \sum_{i=1}^{n}(T_i R_i T_i^T)^{-1} = \sum_{i=1}^{n} T_i R_i^{-1} T_i^T\,, \tag{48}$$

from which we obtain

$$P_{\text{global}}^{-1} P_i^r = 2a_2(r_i - a_1) T_i R_i^{-2} \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} T_i^T P_{\text{global}} \tag{49}$$

$$P_{\text{global}}^{-1} P_i^\theta = \big(A_i + A_i^T\big) P_{\text{global}} \tag{50}$$

with

$$A_i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} T_i R_i^{-1} T_i^T\,. \tag{51}$$

We implement a decentralized version of the resulting gradient control law (42) as follows. Each agent runs an average consensus estimator (such as the PI estimator described in Section III-D) with local matrix input $nT_i R_i^{-1} T_i^T$ (for a total of 3 scalar estimators due to the symmetry of this 2×2 matrix), but with the unknown quantities r_i and θ_i replaced by the measurements

$$r_i \approx |p_i - z_i|\,, \qquad \theta_i \approx \angle(p_i - z_i)\,. \tag{52}$$

The inverse of the output of this estimator is P_i, the local estimate of P_{global}. Each agent runs a second average consensus estimator with local vector input $nT_i R_i^{-1} T_i^T z_i$ (for a total of 2 scalar estimators), again with the replacements (52). The output of this second estimator, when multiplied by P_i, yields $\hat{x}_i$, the local estimate of $\hat{x}_{\text{global}}$. We now evaluate the expressions (41), (49), and (50) by replacing P_{global} with P_i and using the following filtered versions of the replacements (52):

$$r_i \approx |p_i - \hat{x}_i|\,, \qquad \theta_i \approx \angle(p_i - \hat{x}_i)\,. \tag{53}$$

These replacements lead to the decentralized version of the control law (42) with gradients (45) and (46). This implementation assumes that the total number n of agents in the swarm and the sensor model parameters a_0, a_1, a_2, and α are known to each agent.

C. Kalman Filter Approach

The approach in Section IV-B fuses sensor readings from current measurements only. To make use of past measurements as well, we can adopt a Kalman filter approach to defining $\hat{x}_{\text{global}}$ and P_{global}. We begin with a linear target model

$$\dot{x}_t = F x_t + G u_t + w\,, \tag{54}$$

where u_t is an exogenous input and w is a continuous-time Gaussian noise with zero mean and covariance matrix Q. We consider the centralized Kalman filter

$$\dot{P}_{\text{global}} = F P_{\text{global}} + P_{\text{global}} F^T + Q - n P_{\text{global}} C P_{\text{global}} \tag{55}$$

$$\dot{\hat{x}}_{\text{global}} = F\hat{x}_{\text{global}} + G u_t + n P_{\text{global}}(y - C\hat{x}_{\text{global}})\,, \tag{56}$$

where C and y are the fused measurements

$$C = \frac{1}{n}\sum_{i=1}^{n} T_i R_i^{-1} T_i^T\,, \qquad y = \frac{1}{n}\sum_{i=1}^{n} T_i R_i^{-1} T_i^T z_i \tag{57}$$

and initial conditions are given by the one-time measurements (47) and (48). The partial derivatives in (44) are the solutions to the differential equations

$$\begin{aligned}\dot{P}_i^r &= F P_i^r + P_i^r F^T - n P_i^r C P_{\text{global}} - n P_{\text{global}} C P_i^r \\ &\quad + 2a_2(r_i - a_1) P_{\text{global}} T_i R_i^{-2} \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} T_i^T P_{\text{global}}\end{aligned} \tag{58}$$

$$\begin{aligned}\dot{P}_i^\theta &= F P_i^\theta + P_i^\theta F^T - n P_i^\theta C P_{\text{global}} - n P_{\text{global}} C P_i^\theta \\ &\quad + P_{\text{global}}\big(A_i + A_i^T\big) P_{\text{global}}\end{aligned} \tag{59}$$

with initial conditions calculated according to the one-time measurements (49) and (50).

We implement a decentralized version of the resulting gradient control law (42) as follows. Each agent runs two average consensus estimators, one with local matrix input $T_i R_i^{-1} T_i^T$ and local output C_i, and the other with local vector input $T_i R_i^{-1} T_i^T z_i$ and local output y_i. Each agent also maintains estimates P_i and $\hat{x}_i$ of P_{global} and $\hat{x}_{\text{global}}$ (respectively) by means of the differential equations

$$\dot{P}_i = F P_i + P_i F^T + Q - n P_i C_i P_i \tag{60}$$

$$\dot{\hat{x}}_i = F\hat{x}_i + G u_t + n P_i(y_i - C_i\hat{x}_i) \tag{61}$$

with initial conditions

$$P_i(0) = (T_i R_i T_i^T)(0)\,, \qquad \hat{x}_i(0) = z_i(0)\,. \tag{62}$$

Finally, each agent maintains local copies of the gradients P_i^r and P_i^θ (which we again name P_i^r and P_i^θ with a slight abuse of notation) by means of the differential equations

$$\begin{aligned}\dot{P}_i^r &= F P_i^r + P_i^r F^T - n P_i^r C_i P_i - n P_i C_i P_i^r \\ &\quad + 2a_2(r_i - a_1) P_i T_i R_i^{-2} \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} T_i^T P_i\end{aligned} \tag{63}$$

$$\begin{aligned}\dot{P}_i^\theta &= F P_i^\theta + P_i^\theta F^T - n P_i^\theta C_i P_i - n P_i C_i P_i^\theta \\ &\quad + P_i\big(A_i + A_i^T\big) P_i\end{aligned} \tag{64}$$

with initial conditions given by (49) and (50) but with $P_i(0)$ replacing $P_{\text{global}}(0)$. In all of these equations we use the replacements (53), and we arrive at an implementable version of the local controller (42). This implementation assumes that F, G, Q, u_t, n, and the sensor model parameters a_0, a_1, a_2, and α are known to each agent.

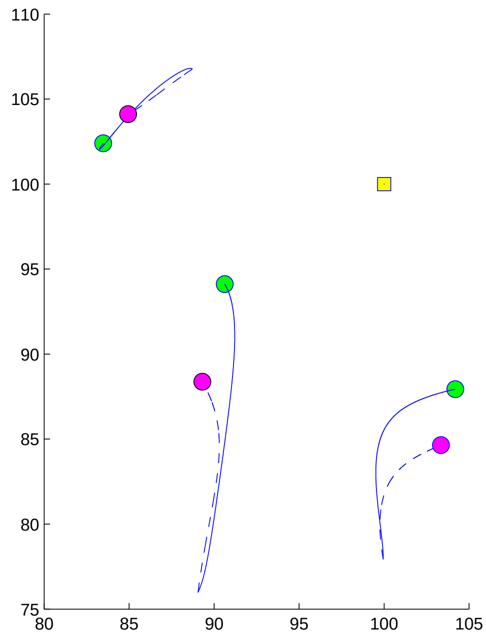


Fig. 9. Trajectories of sensors with stationary target at $(100, 100)$. The solid lines denote the Kalman filter scheme and the dashed lines denote the one-time measurement scheme.

D. Simulation Results

We use three mobile sensors starting from $(88.73, 106.76)$, $(89.05, 75.98)$, $(99.94, 77.93)$ and a stationary target sitting at $(100, 100)$. The sensor model parameters are $a_0 = 0.3528$, $a_1 = 15.625$, $a_2 = 0.0008$, and $\alpha = 5$. The communication radius is set at $r = 50$, and we choose a controller gain of $\Gamma = 20I$. The PI estimator is implemented for every average consensus estimation task. Figure 9 shows the actual trajectories of the sensors. In Figure 10 we compare the performance of these decentralized algorithms with each other and with the centralized versions. In both cases, the decentralized schemes recover the results of their centralized counterparts after an initial transient.

Figure 11 compares the performance of static and mobile sensor fusion schemes. Sensors start from the same positions, and in this simulation we changed the parameter a_2 of the sensor model to 0.5 to increase the spatial influence on the measurement noise level. We see that the moving sensors more quickly obtain accurate estimates of the target position.

We used the PI estimator in this application because the simpler high-pass estimator from Section III-C does not filter noise at high frequencies. Possible alternatives to the PI estimator may include the low-pass and band-pass consensus estimators proposed in [32], although further analysis is needed to determine their efficacy for this application.

We do not yet have stability results for the schemes proposed in Sections IV-B and IV-C, and it may well be that additional terms are needed in the local controller (42) to guarantee stability (as was the case in the formation control problem of Section III). Nevertheless, our simulations demonstrate that stable cooperative target localization is achieved.

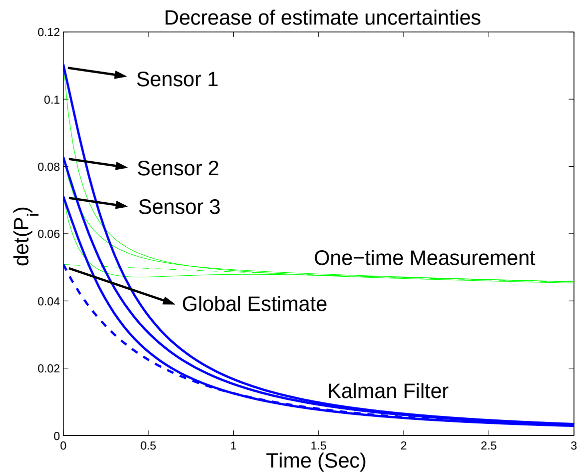


Fig. 10. Comparison of the individual belief uncertainty matrices P_i. The centralized versions P_{global} for each scheme are shown as dotted lines.

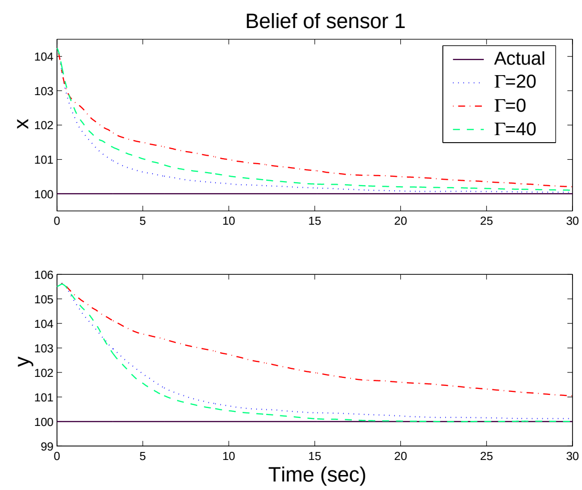


Fig. 11. Comparison between static sensors (Γ=0) and mobile sensors ($\Gamma = 20I, 40I$).

V. SUMMARY AND FUTURE WORK

We have presented a framework for decentralized estimation and control to achieve desired “emergent” behaviors of multi-agent systems. We have given two instantiations based on dynamic average consensus estimators: control of formation moments and cooperative target localization. For the formation control problem, we have provided motion control algorithms, estimator algorithms, and small-gain conditions guaranteeing convergence as a function of the aggressiveness of the motion controls and the communication network Laplacian. Simulations confirm that our algorithms generate the desired swarm behavior, often even when our assumptions are violated or when certain controller nonlinearities are neglected.

This work can be extended in several directions. In the formation control problem, the gradient controllers can be modified to include auxiliary objectives without affecting the convergence proofs. For example, we have broad freedom in choosing the damping matrix B; it could be made configuration-dependent for agent-agent and agent-obstacle avoidance or for maintaining network connectivity. Further work is needed to design these terms to guarantee their objectives. We could also consider the tracking problem in which the desired formation vector $f^\star$ changes with time, and problems where agents are nonholonomic or underactuated. Next, although we expect the PI estimator to have better sensor noise attenuation properties than the high-pass estimator, further analysis is needed. Similarly, we need to investigate the effects of noise and time delay in the communication channels between the robots. Next, we can develop discrete-time versions of the algorithms or perform a hybrid dwell-time analysis of the switching topology, in order to derive averaged small-gain conditions. We may also consider adaptive algorithms for adjusting the estimator gains to improve estimation convergence rates, allowing more aggressive motion controls.

Finally, the framework we have adopted is quite general, but our technical results apply to very specific problems. We would like to identify subclasses of problems for which the four-step design methodology can be partially or completely automated.

APPENDIX

A. Subanalytic sets and functions

Let M be a C^ω manifold of dimension m.[7] For each open set $U \subset M$, let $C^\omega(U)$ denote the ring of real-valued, C^ω functions on U, and let $\mathfrak{S}(U)$ denote the Boolean algebra of subsets of U generated by all sets $\{x \in U : g(x) > 0\}$ such that $g \in C^\omega(U)$.

A set $S \subset M$ is called *semianalytic* when each point in M has an open neighborhood $U \subset M$ such that $S \cap U \in \mathfrak{S}(U)$. Note that if ϕ is a coordinate mapping for M and $B \subset \mathbb{R}^m$ is an open ball such that $\mathrm{cl}(B) \subset \mathrm{Im}(\phi)$, then $\phi^{-1}(B)$ is a relatively compact, semianalytic, open subset of M; in particular, the collection of all such subsets of M form a base for the topology for M.

A set $S \subset M$ is called *subanalytic* when each point in M has an open neighborhood $U \subset M$ such that $S \cap U$ is the projection of a relatively compact semianalytic set (i.e., there is a C^ω manifold N and a relatively compact semianalytic set $A \subset M \times N$ such that $S \cap U = \pi(A)$, where $\pi : M \times N \to M$ is the natural projection). Every semianalytic set is subanalytic, but not conversely.

Both semianalyticity and subanalyticity are local properties: if a set $S \subset M$ is such that every point in M has an open neighborhood $U \subset M$ such that $S \cap U$ has the property, then the set S itself has the property.[8] Also, the collections of semianalytic and subanalytic sets are closed under the operations of taking finite unions, finite intersections, complements, Cartesian products, closures, and interiors [3], [18]. Finally, the image of a relatively compact subanalytic set by a subanalytic function is subanalytic [3] (in particular, the projection of a relatively compact subanalytic set is subanalytic).

[7] More generally, M can be the disjoint union of manifolds of different dimensions, such as the set $\mathfrak{P}$ of swarm configurations from Section III-A.

[8] To prove this for semianalyticity, we make use of the following fact: suppose $U, V \subset M$ are open with $U \subset V$; then because the restriction of every member of $C^\omega(V)$ to U is a member of $C^\omega(U)$, it follows that $S \cap U \in \mathfrak{S}(U)$ for every $S \in \mathfrak{S}(V)$.

Let M and N be C^ω manifolds. A function $f : D \to N$ defined on a set $D \subset M$ is called *subanalytic* when its graph is a subanalytic subset of $M \times N$. Subanalytic functions represent a specific class of "piecewise analytic" functions; in particular, every C^ω function defined on an open set D is subanalytic. In general, the domain D of a subanalytic function f need not be subanalytic.[9] A class of subanalytic functions having subanalytic domains are the locally bounded ones: we say that a function $f : D \to N$ is *locally bounded* when $f(D \cap U)$ is relatively compact in N for every relatively compact $U \subset M$. In particular, if f is continuous and D is closed, then f is locally bounded. Note that this definition of local boundedness depends on the manifold M as well as the domain D: the function $f(x) = 1/x$ defined on the open interval $D = (0, 1)$ is locally bounded when $M = D$ but not when $M = \mathbb{R}$.

Lemma 5: Let M and N be C^ω manifolds, let $D \subset M$, let $f : D \to N$ be subanalytic, let $V \subset N$ be subanalytic, and suppose either that f is locally bounded or that V is relatively compact. Then $f^{-1}(V)$ is subanalytic.

Proof: Let $F \subset M \times N$ be the graph of the function f; then $f^{-1}(V) = \pi(F \cap (M \times V))$, where $\pi : M \times N \to M$ denotes the natural projection. Fix $x \in M$ and let $U \subset M$ be a relatively compact, semianalytic, open neighborhood of x. The set $H = F \cap (U \times V)$ is subanalytic (being comprised of intersections and products of subanalytic sets) and relatively compact (which is obvious when V is relatively compact, and true also if f is locally bounded because $H \subset U \times f(D \cap U)$ and U is relatively compact). Hence $f^{-1}(V) \cap U = \pi(H)$ is subanalytic, and because subanalyticity is a local property, we conclude that $f^{-1}(V)$ itself is subanalytic. ∎

Lemma 6: Let L, M, and N be C^ω manifolds, and suppose $f : D \to N$ and $g : C \to M$ are subanalytic functions defined on sets $D \subset M$ and $C \subset L$ (respectively) with $\mathrm{Im}(g) \subset D$. If either f is proper or g is locally bounded, then the composition $f \circ g$ is subanalytic.

Proof: Following the proof of [4, Proposition 2.2.6], let $F \subset M \times N$ and $G \subset L \times M$ be the graphs of f and g, respectively. Define $H = (G \times N) \cap (L \times F)$; then $\pi(H)$ is the graph of $f \circ g$, where $\pi : L \times M \times N \to L \times N$ denotes the natural projection. Fix $x \in L$ and $y \in N$, let $U_x \subset L$ and $U_y \subset N$ be relatively compact, semianalytic, open neighborhoods of x and y (respectively), and define the semianalytic sets $U = U_x \times U_y$ and $T = U_x \times M \times U_y$. We next show that $H \cap T$ is relatively compact. We can bound $H \cap T$ in two different ways as follows:

$$H \cap T \subset (L \times F) \cap T \subset U_x \times \big[F \cap (M \times U_y)\big]\,, \tag{65}$$

$$H \cap T \subset (G \times N) \cap T \subset \big[G \cap (U_x \times M)\big] \times U_y\,. \tag{66}$$

Furthermore, we have

$$\begin{aligned} F \cap (M \times U_y) &\subset f^{-1}(U_y) \times U_y \\ &\subset f^{-1}(\mathrm{cl}(U_y)) \times \mathrm{cl}(U_y)\,, \end{aligned} \tag{67}$$

$$G \cap (U_x \times M) \subset U_x \times g(C \cap U_x)\,. \tag{68}$$

[9] For example, let $M = N = \mathbb{R}$ and let f be a bijection from the rationals $\mathbb{Q} \subset \mathbb{R}$ to the integers $\mathbb{Z} \subset \mathbb{R}$. Then each point in the graph of f is isolated (and hence f is subanalytic), but the domain $D = \mathbb{Q}$ of f is not subanalytic.

If f is proper, then the set on the right-hand side of (67) is compact, and we conclude from (65) that $H \cap T$ is relatively compact. Similarly, if g is locally bounded, then the set on the right-hand side of (68) is relatively compact, and we conclude from (66) that $H \cap T$ is relatively compact. Because $H \cap T$ is also subanalytic with $\pi(H) \cap U = \pi(H \cap T)$, we conclude that $\pi(H) \cap U$ is subanalytic. Finally, because subanalyticity is a local property, we conclude that $\pi(H)$ is subanalytic. ■

Corollary 7: Let D be a subset of a C^ω manifold. If functions $f_1, f_2 : D \to \mathbb{R}$ are subanalytic and locally bounded, then their sum $f_1 + f_2$, difference $f_1 - f_2$, and product $f_1 f_2$ are subanalytic and locally bounded. If in addition $f_2 \neq 0$ on D, then their ratio f_1/f_2 is subanalytic.

Proof: Define $g : D \to \mathbb{R}^2$ as $g(x) = (f_1(x), f_2(x))$; then g is subanalytic and locally bounded, so by Lemma 6 its composition with any real-valued subanalytic function defined on $\mathrm{Im}(g) \subset \mathbb{R}^2$ is subanalytic. ■

Lemma 8: Let $M_1 \subset \mathbb{R}$ be open, let M_2 be a C^ω manifold, let $f : M_1 \times M_2 \to \mathbb{R}$ be a C^1 subanalytic function, and let $f_1 : M_1 \times M_2 \to \mathbb{R}$ be the C^0 partial derivative of f with respect to its first argument. Then f_1 is subanalytic.

Proof: Following the proof of [4, Proposition 2.9.1], we define $h : D \to \mathbb{R}$ on the open semianalytic set

$$D = \big\{(x, y, z) \in M_1 \times M_1 \times M_2 \,:\, x \neq y\big\} \tag{69}$$

as $h(x, y, z) = (f(x, z) - f(y, z))/(x - y)$ (which is subanalytic from Corollary 7). Because f is C^1, this function h has a continuous extension $\bar{h} : M_1 \times M_1 \times M_2 \to \mathbb{R}$. The graph of $\bar{h}$ is the closure of the graph of h, and it follows that $\bar{h}$ is subanalytic. Now $f_1 = \bar{h} \circ g$, where g is the mapping taking $(x, z) \in M_1 \times M_2$ to $(x, x, z) \in M_1 \times M_1 \times M_2$, and it follows from Lemma 6 that f_1 is subanalytic. ■

The following is a consequence of Lemmas 5 and 8:

Corollary 9: Let $f : M \to \mathbb{R}$ be a C^1 subanalytic function on a C^ω manifold M. Then the set $\mathrm{Crit}(f) \subset M$ of all critical points of f is subanalytic.

Theorem 10: If $f : M \to \mathbb{R}$ is a C^1 subanalytic function on a C^ω manifold M, then f is locally constant on $\mathrm{Crit}(f)$.

Proof: The set $\mathrm{Crit}(f)$ is subanalytic by Corollary 9. Because subanalytic sets admit stratifications [17], we can partition $\mathrm{Crit}(f)$ into a locally finite family $\mathcal{S}$ of connected C^ω submanifolds of M (called strata) such that if $P, Q \in \mathcal{S}$ are such that $Q \cap \mathrm{cl}(P) \neq \varnothing$ then $Q \subset \mathrm{cl}(P)$. We want to show that each point $p \in \mathrm{Crit}(f)$ has an open neighborhood $N_p \subset M$ such that f is constant on $N_p \cap \mathrm{Crit}(f)$. Let U be an open neighborhood of p which meets only a finite number of strata $P_1, \ldots, P_n \in \mathcal{S}$. We let these n strata represent the n nodes of a graph G such that P_i and P_j are adjacent whenever they are not separated,[10] or equivalently whenever either $P_i \subset \mathrm{cl}(P_j)$ or $P_j \subset \mathrm{cl}(P_i)$. It follows from Stokes' theorem that f is constant on each stratum P_i and thus also on every pair of G-adjacent strata (P_i, P_j). Let i be such that $p \in P_i$, and let G_p denote the connected component of G containing node P_i; then f is constant on the union of the nodes in G_p. Finally, choose an open neighborhood $N_p \subset U$ of p such that N_p does not meet any node in $G \setminus G_p$. ■

[10] Recall that two subsets $A, B \subset \mathcal{X}$ of a topological space $\mathcal{X}$ are said to be *separated* when $A \cap \mathrm{cl}(B) = \mathrm{cl}(A) \cap B = \varnothing$.

B. Proof of Theorem 2

Let M denote the subspace of $R^{m\times(m+1)}$ comprised of all matrices of the form $[x \quad X]$, where $x \in \mathbb{R}^m$ and X is a symmetric $m \times m$ matrix. Let $\mathcal{V} : M \to \mathbb{R}^\ell$ denote the invertible linear map given by

$$\mathcal{V}\big([x \quad X]\big) \triangleq \begin{bmatrix} x \\ \mathrm{triu}(X) \end{bmatrix}. \tag{70}$$

First suppose that $f^\star$ has the special form

$$f^\star = \mathcal{V}\big([0 \quad \Delta]\big) \tag{71}$$

for some diagonal matrix $\Delta \geqslant 0$. We now show that any diagonal $\Gamma > 0$ works in this case. Given a swarm configuration $p \in \mathfrak{P}$, we let $n = n(p)$ be such that $p \in \mathbb{R}^{mn}$, and we define $P \triangleq [p_1 \ \ldots \ p_n] \in \mathbb{R}^{m\times n}$ so that $p = \mathrm{vec}(P)$ (where $\mathrm{vec}(\cdot)$ is the invertible linear map which stacks the columns of a matrix to produce a vector). For convenience we introduce $q = \mathrm{vec}(Q)$, where $Q = P^T = [q_1 \ \ldots \ q_m] \in \mathbb{R}^{n(p)\times m}$. In these coordinates, the function $f(p)$ becomes

$$f(p) = \frac{1}{n(p)} \begin{bmatrix} Q^T\mathbf{1} \\ \mathrm{triu}(Q^TQ) \end{bmatrix}. \tag{72}$$

Computing $\mathcal{J}_q f$, the q-Jacobian of f, we obtain

$$\mathcal{J}_q f = \frac{1}{n(p)} \begin{bmatrix}
\mathbf{1}^T & 0 & 0 & \ldots & 0 & 0 \\
0 & \mathbf{1}^T & 0 & \ldots & 0 & 0 \\
0 & 0 & \mathbf{1}^T & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \mathbf{1}^T & 0 \\
0 & 0 & 0 & \ldots & 0 & \mathbf{1}^T \\
2q_1^T & 0 & 0 & \ldots & 0 & 0 \\
0 & 2q_2^T & 0 & \ldots & 0 & 0 \\
0 & 0 & 2q_3^T & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 2q_{m-1}^T & 0 \\
0 & 0 & 0 & \ldots & 0 & 2q_m^T \\
q_2^T & q_1^T & 0 & \ldots & 0 & 0 \\
0 & q_3^T & q_2^T & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & q_{m-1}^T & q_{m-2}^T & 0 \\
0 & 0 & \ldots & 0 & q_m^T & q_{m-1}^T \\
q_3^T & 0 & q_1^T & 0 & \ldots & 0 \\
0 & q_4^T & 0 & q_2^T & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & q_m^T & 0 & q_{m-2}^T \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
q_m^T & 0 & 0 & \ldots & 0 & q_1^T
\end{bmatrix}.$$

If we define

$$z = \Gamma\big[f(p) - f^\star\big], \tag{73}$$

then we can write the q-gradient of J in (8) as

$$\nabla_q J(p) = 2\big[\mathcal{J}_q f\big]^T z\,. \tag{74}$$

We write $z = [z_1 \ \dots \ z_\ell]^T$, $\Gamma = \mathrm{diag}(\gamma_1, \dots, \gamma_\ell)$ with each $\gamma_i > 0$, and $\Delta = \mathrm{diag}(\delta_1, \dots, \delta_m)$ with each $\delta_i \geqslant 0$. Now suppose p is such that $\nabla J(p) = 0$ but $z \neq 0$; then also $\nabla_q J(p) = 0$, and in particular

$$\begin{aligned} 0 &= \big[q_1^T \ 0 \ \dots \ 0\big]\big[\mathcal{J}_q f\big]^T z & (75)\\ &= \frac{1}{n^2(p)}\big[\gamma_1(\mathbf{1}^T q_1)^2 + 2\gamma_{m+1}(q_1^T q_1)\big(q_1^T q_1 - n(p)\delta_1\big) \\ &\qquad + \gamma_{2m+1}(q_2^T q_1)^2 + \dots + \gamma_\ell(q_m^T q_1)^2\big]\,. & (76)\end{aligned}$$

This implies $q_1^T q_1 \leqslant n(p)\delta_1$, and in a similar manner we can show that $q_i^T q_i \leqslant n(p)\delta_i$ for $1 \leqslant i \leqslant m$. Furthermore, because $z \neq 0$, there exists some j such that $q_j^T q_j < n(p)\delta_j$, and for simplicity we assume $j = 1$ (the other cases are similar). Let $v \in \mathbb{R}^{mn(p)}$ be a constant vector, and define

$$F = v^T \nabla_q J(p) = 2w^T z\,, \quad \text{where} \quad w = \big[\mathcal{J}_q f\big]v\,. \tag{77}$$

If we let $\mathcal{H}_q J(p)$ denote the q-Hessian of J at p, then

$$\big[\nabla_q F\big]^T v = v^T\big[\mathcal{H}_q J(p)\big]v \ = \ 2w^T\Gamma w + 2z^T\big[\mathcal{J}_q w\big]v\,. \tag{78}$$

Now $\nabla_q J(p) = 0$ and $z \neq 0$ imply $\mathrm{rank}[\mathbf{1} \ \ Q] < m+1$; thus because $n(p) \geqslant m+1$ there exists a nonzero $u \in \mathbb{R}^{n(p)}$ such that $u^T[\mathbf{1} \ \ Q] = 0$. Choosing $v = [u^T \ \ 0 \ \ \dots \ \ 0]^T$, we see that $w = 0$ and thus

$$\begin{aligned} v^T\big[\mathcal{H}_q J(p)\big]v &= 2z^T\big[\mathcal{J}_q w\big]v \ = \ 2z^T\frac{\partial w}{\partial q_1}\,u \\ &= \frac{4}{n(p)}\,z_{m+1}\cdot(u^T u) \\ &= \frac{4}{n^2(p)}\,\gamma_{m+1}(u^T u)\big(q_1^T q_1 - n(p)\delta_1\big) \\ &< 0\,. \end{aligned} \tag{79}$$

Therefore the Hessian of J has at least one strictly negative eigenvalue at p, and we conclude that p cannot be a local minimum of J.

To complete the proof, we show the existence of a computable coordinate change *which is independent of* $n(p)$ and is such that any $f^\star \in f(\mathcal{D})$ takes the special form (71) in the new coordinates. For each n, define the map $K_n : \mathbb{R}^{m\times n} \to M$ as

$$K_n(X) \triangleq \frac{1}{n}\big[X\mathbf{1}_n \quad XX^T\big] \tag{80}$$

for $X \in \mathbb{R}^{m\times n}$. Then because

$$\sum_{i=1}^{n}\big[p_i \quad p_i p_i^T\big] = \big[P\mathbf{1}_n \quad PP^T\big]\,, \tag{81}$$

we can write the moment vector $f(p)$ in (11) as

$$f(p) = \big(\mathcal{V}\circ K_{n(p)}\circ \mathrm{vec}^{-1}\big)(p)\,. \tag{82}$$

Given $f^\star \in f(\mathcal{D})$, let $\nu \geqslant m+1$ and $p^\star \in \mathbb{R}^{m\nu}$ be such that $f(p^\star) = f^\star$, and define $P^\star = \mathrm{vec}^{-1}(p^\star) \in \mathbb{R}^{m\times\nu}$. We fix $S \in \mathrm{Orth}(\mathbf{1}_\nu)$ and write the SVD of the matrix $P^\star S$ as $P^\star S = U\Sigma V^T$, where $U \in \mathbb{R}^{m\times m}$ is an orthogonal matrix. Using U, for each n we construct the invertible affine transformation $J_n : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ as follows:

$$J_n(X) \triangleq U^T\big[X - \frac{1}{\nu}\,P^\star\mathbf{1}_\nu\mathbf{1}_n^T\big]\,, \tag{83}$$

where $X \in \mathbb{R}^{m\times n}$. This mapping has the property that

$$(K_\nu \circ J_\nu)(P^\star) = \frac{1}{\nu}\big[0 \quad \Sigma\Sigma^T\big]\,. \tag{84}$$

Next we define $T : M \to M$ as

$$T\big([x \quad X]\big) \triangleq \big[T_1(x) \quad T_2(x, X)\big] \tag{85}$$

where $x \in \mathbb{R}^m$, $X \in \mathbb{R}^{m\times m}$ is symmetric, and

$$\begin{aligned} T_1(x) &\triangleq U^T\big[x - \frac{1}{\nu}P^\star\mathbf{1}_\nu\big]\,, & (86)\\ T_2(x, X) &\triangleq U^T\big[X - \frac{1}{\nu}\,P^\star\mathbf{1}_\nu x^T - \frac{1}{\nu}\,x\mathbf{1}_\nu^T(P^\star)^T \\ &\qquad + \frac{1}{\nu^2}P^\star\mathbf{1}_\nu\mathbf{1}_\nu^T(P^\star)^T\big]U\,. & (87)\end{aligned}$$

It is clear that T is an invertible affine map, and it is straightforward to verify that $K_n \circ J_n \equiv T \circ K_n$ for any n. We define $G : \mathfrak{P} \to \mathfrak{P}$ and $W : \mathbb{R}^\ell \to \mathbb{R}^\ell$ as

$$\begin{aligned} G(p) &\triangleq \big(\mathrm{vec}\circ J_{n(p)}\circ \mathrm{vec}^{-1}\big)(p) & (88)\\ W(x) &\triangleq \big(\mathcal{V}\circ T\circ \mathcal{V}^{-1}\big)(x) & (89)\end{aligned}$$

for $p \in \mathfrak{P}$ and $x \in \mathbb{R}^\ell$. These maps G and W are such that $f\circ G \equiv W\circ f$, and in particular we have

$$\big(f\circ G\big)(p^\star) = W(f^\star) = \frac{1}{\nu}\,\mathcal{V}\big(\big[0 \quad \Sigma\Sigma^T\big]\big)\,. \tag{90}$$

To summarize, we have the following commutative diagram:

$$\begin{array}{ccccccc} \mathfrak{P} & \xrightarrow{\ f\ } & \mathbb{R}^\ell & \xrightarrow{\ \mathcal{V}^{-1}\ } & M & \xleftarrow{\ K_n\ } & \mathbb{R}^{m\times n} \\ G\big\downarrow & & \big\downarrow W & & \big\downarrow T & & \big\downarrow J_n \\ \mathfrak{P} & \xrightarrow{\ f\ } & \mathbb{R}^\ell & \xleftarrow{\ \mathcal{V}\ } & M & \xleftarrow{\ K_n\ } & \mathbb{R}^{m\times n} \end{array} \tag{91}$$

Because W is an invertible affine map, there exist an invertible matrix $A \in \mathbb{R}^{\ell\times\ell}$ and $b \in \mathbb{R}^\ell$ be such that $W(x) = Ax + b$ for all $x \in \mathbb{R}^\ell$. Thus for $\Gamma > 0$,

$$\begin{aligned} \big[f(G(p)) - W(f^\star)\big]^T(A^T)^{-1}\Gamma A^{-1}\big[f(G(p)) - W(f^\star)\big] \\ = \big[f(p) - f^\star\big]^T\Gamma\big[f(p) - f^\star\big] \end{aligned} \tag{92}$$

for all $p \in \mathfrak{P}$. Now because the restriction of G to each set $\mathbb{R}^{mn}$ is invertible, we have $\Gamma \in \mathcal{G}(f^\star, \mathcal{D})$ if and only if $\Gamma_0 \in \mathcal{G}(W(f^\star), \mathcal{D})$, where

$$\Gamma = A^T\Gamma_0 A\,. \tag{93}$$

Thus without loss of generality, we may assume that $f^\star$ has the structure of $W(f^\star)$, namely, that $f^\star$ has the special form (71).

C. *Proof of Theorem 3*

We will need the following technical result:

Lemma 11: Let $\mathcal{X}$ be a topological space, let $\mathcal{Y}$ be a set, let $A \subset \mathcal{X}$, let $B \subset A$ be connected, and let $h : \mathcal{X} \to \mathcal{Y}$. Suppose every $x \in B$ has an open neighborhood $N_x \subset \mathcal{X}$ such that h is constant on $\mathrm{cl}(N_x) \cap A$. Then h is constant on $\mathrm{cl}(N) \cap A$, where $N \triangleq \bigcup\{N_x : x \in B\}$.

Proof: Suppose there exist $x_1, x_2 \in \mathrm{cl}(N) \cap A$ such that $h(x_1) \neq h(x_2)$, and define the following subsets of $\mathcal{X}$:

$$N_1 \triangleq \bigcup \big\{ N_x \,:\, h(\cdot) = h(x_1) \text{ on } \mathrm{cl}(N_x) \cap A \big\}\,, \tag{94}$$

$$N_2 \triangleq \bigcup \big\{ N_x \,:\, h(\cdot) \neq h(x_1) \text{ on } \mathrm{cl}(N_x) \cap A \big\}\,, \tag{95}$$

$$\mathcal{O}_1 \triangleq N_1 \cap B\,, \qquad \mathcal{O}_2 \triangleq N_2 \cap B\,. \tag{96}$$

Then $\mathcal{O}_1$ and $\mathcal{O}_2$ are nonempty, disjoint, open relative to B, and such that $B = \mathcal{O}_1 \cup \mathcal{O}_2$. However, this contradicts the fact that B is connected. ■

Consider the consensus estimators (20)–(21). Defining variables $z_i, e_i \in \mathbb{R}^\ell$ as

$$z_i = \frac{d}{dt}\,\phi(p_i) = \big[\partial\phi(p_i)\big]\dot{p}_i \tag{97}$$

$$e_i = f(p) - y_i\,, \tag{98}$$

we introduce the following $\ell \times n$ matrices:

$$Y = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix} \tag{99}$$

$$H = \begin{bmatrix} \eta_1 & \dots & \eta_n \end{bmatrix} \tag{100}$$

$$\Phi(p) = \begin{bmatrix} \phi(p_1) & \dots & \phi(p_n) \end{bmatrix} \tag{101}$$

$$E = \begin{bmatrix} e_1 & \dots & e_n \end{bmatrix} = \Phi(p)\,\frac{\mathbf{1}\mathbf{1}^T}{n} - Y \tag{102}$$

$$Z = \begin{bmatrix} z_1 & \dots & z_n \end{bmatrix} = \frac{d}{dt}\,\Phi(p)\,. \tag{103}$$

Hence we may write the collection of consensus estimators (20)–(21) in matrix form as

$$\dot{H} = -\gamma H - YL(p) \tag{104}$$

$$Y = H + \Phi(p)\,. \tag{105}$$

We write the complete state of the closed-loop system as either the triple $(p, \dot{p}, H)$, or with the global coordinate change given by (102) and (105), the triple $(p, \dot{p}, E)$. We see from (22) that the derivative of the storage function

$$V(p, \dot{p}) = \dot{p}^T\dot{p} \,+\, n\,J(p) \tag{106}$$

can be written as

$$\dot{V} = \sum_{i=1}^{n} \Big[-\dot{p}_i^T \big[B + B^T\big]\dot{p}_i - z_i^T\big[\Lambda + \Lambda^T\big]z_i + 2z_i^T\Gamma e_i \Big]\,. \tag{107}$$

We observe from (102) and (105) that $E\mathbf{1} \equiv -H\mathbf{1}$ and thus

$$\dot{E}\mathbf{1} = \gamma H\mathbf{1} = 0 \qquad \text{(using } \gamma = 0\text{)}, \tag{108}$$

which means

$$E(t)\mathbf{1} = -H(t_0)\mathbf{1} = 0 \qquad \text{(using } \eta_i(t_0) = 0\text{)} \tag{109}$$

for any $t \geqslant t_0$. Because $ES \equiv -YS$, we can write

$$\dot{E}S = YL(p)S - ZS = -ESL^\star(p) - ZS\,. \tag{110}$$

Also because $E\mathbf{1} \equiv 0$ we have $EE^T \equiv ESS^TE^T$, so

$$\begin{aligned} \frac{d}{dt}\,EE^T &= \dot{E}SS^TE^T + ESS^T\dot{E}^T \\ &= -2ESL^\star(p)S^TE^T - ZSS^TE^T - ESS^TZ^T \\ &\leqslant -\varepsilon EE^T + \frac{1}{\varepsilon}ZSS^TZ^T \\ &\leqslant -\varepsilon EE^T + \frac{1}{\varepsilon}ZZ^T\,. \end{aligned} \tag{111}$$

Defining the storage function $U = \mathrm{Tr}(EE^T)$ we see that

$$\dot{U} \leqslant -\varepsilon U + \frac{1}{\varepsilon}\,\mathrm{Tr}(ZZ^T) = \sum_{i=1}^{n}\Big[-\varepsilon|e_i|^2 + \frac{1}{\varepsilon}|z_i|^2\Big]\,. \tag{112}$$

Furthermore, using the fact that

$$2z_i^T\Gamma e_i \leqslant \frac{1}{2}\,z_i^T\big[\Lambda + \Lambda^T\big]z_i + 2e_i^T\Gamma\big[\Lambda + \Lambda^T\big]^{-1}\Gamma e_i\,, \tag{113}$$

we can bound (107) from above as

$$\dot{V} \leqslant \sum_{i=1}^{n}\Big[-\dot{p}_i^T\big[B + B^T\big]\dot{p}_i - \frac{1}{2}\,z_i^T\big[\Lambda + \Lambda^T\big]z_i + 2e_i^T\Gamma\big[\Lambda + \Lambda^T\big]^{-1}\Gamma e_i\Big]\,. \tag{114}$$

To combine the storage functions V and U, we first use (26) to choose $\mu > 0$ such that

$$\Gamma < \mu I < \frac{\varepsilon}{2}\,\big[\Lambda + \Lambda^T\big]\,. \tag{115}$$

Upon taking inverses and then multiplying by Γ from the left and the right, we obtain

$$\frac{2}{\varepsilon}\,\Gamma\big[\Lambda + \Lambda^T\big]^{-1}\Gamma < \Gamma\,. \tag{116}$$

In particular, (115) and (116) imply the existence of a scalar constant $\nu > 0$ such that

$$\frac{1}{2}\big[\Lambda + \Lambda^T\big] - \frac{\mu}{\varepsilon}\,I \geqslant \nu I \tag{117}$$

$$\mu\varepsilon I - 2\Gamma\big[\Lambda + \Lambda^T\big]^{-1}\Gamma \geqslant \nu I\,. \tag{118}$$

We then define $\Upsilon(p, \dot{p}, E) = V + \mu U$ and use (112), (114), (117), and (118) to obtain

$$\dot{\Upsilon} \leqslant \sum_{i=1}^{n}\Big[-\dot{p}_i^T\big[B + B^T\big]\dot{p}_i - \nu|z_i|^2 - \nu|e_i|^2\Big]\,. \tag{119}$$

In particular, $\Upsilon(t)$ is nonincreasing along trajectories in forward time. Because $J(p)$ is proper, Υ is a proper function of the states p, $\dot{p}$, and E, and we can conclude that all signals are bounded in forward time. By LaSalle's theorem we further conclude that every trajectory converges to its nonempty, compact, connected positive limit set L^+, and that every point in L^+ is an equilibrium point of the form $(p, \dot{p}, E) = (\bar{p}, 0, 0)$ for some $\bar{p} \in \mathbb{R}^{mn}$ such that

$$\big[\partial\phi(\bar{p}_i)\big]^T\Gamma\big[f^\star - f(\bar{p})\big] = 0 \tag{120}$$

for every i, or equivalently such that $\nabla J(\bar{p}) = 0$. It follows that Υ and thus also J are constant on every positive limit set. In particular, any positive limit set containing one bad equilibrium must contain only bad equilibria.

Next suppose ϕ is subanalytic, suppose $\Gamma \in \mathcal{G}(f^\star, \mathcal{D})$, and suppose a positive limit set L^+ consists of bad equilibria. We can write L^+ as the product $L^+ = L_0^+ \times \{0\} \times \{0\}$, where $L_0^+ \subset \mathcal{D} \cap \mathrm{Crit}(J)$. Because ϕ is subanalytic, so is J, and thus by Theorem 10, J is locally constant on $\mathrm{Crit}(J)$. Hence every point $p \in L_0^+$ has an open neighborhood N_p such that J is constant on the set $\mathrm{cl}(N_p) \cap \mathrm{Crit}(J)$. Define the open set $N \triangleq \bigcup\{N_p : p \in L_0^+\}$; then the set $\mathcal{O} = N \times \mathbb{R}^{mn} \times \mathbb{R}^{\ell \times n}$ is an open neighborhood of L^+. Also, it follows from Lemma 11 that J is constant on $\mathrm{cl}(N) \cap \mathrm{Crit}(J)$. Let $\mathcal{U}$ be any open set such that $L^+ \cap \mathcal{U} \neq \varnothing$, and fix $(\bar{p}, 0, 0) \in L^+ \cap \mathcal{U}$. By assumption $\bar{p}$ is not a local minimum of J, which means there exists a point $(p_0, 0, 0) \in \mathcal{U}$ such that $J(p_0) < J(\bar{p})$ and therefore also $\Upsilon(p_0, 0, 0) < \Upsilon(\bar{p}, 0, 0)$. Let $L_1^+ \times \{0\} \times \{0\}$ denote the positive limit set of the trajectory starting from the state $(p_0, 0, 0)$. Then J is constant on L_1^+, and because Υ is nonincreasing along trajectories, the value of J on L_1^+ is strictly less than its value on $\mathrm{cl}(N) \cap \mathrm{Crit}(J)$. Thus because $L_1^+ \subset \mathrm{Crit}(J)$ we have $L_1^+ \cap \mathrm{cl}(N) = \varnothing$, and it follows that the trajectory starting from $(p_0, 0, 0)$ eventually leaves the open neighborhood $\mathcal{O}$ of L^+ forever. We conclude that L^+ is strongly unsteady.

D. Proof of Theorem 4

The derivative of the storage function (106) is

$$\dot{V} = \sum_{i=1}^{n} \Big[-\dot{p}_i^T \big[B + B^T\big] \dot{p}_i - 2c\zeta(p_i)|\dot{p}_i|^2 - z_i^T \big[\Lambda + \Lambda^T\big] z_i + 2 z_i^T \Gamma e_i \Big] \tag{121}$$

with z_i and e_i as in (97)–(98). We may write the collection of PI estimators (32)–(34) in matrix form as

$$\dot{Y} = -Y\big[\gamma I + L_P(p)\big] + W L_I(p) + \gamma \Phi(p) \tag{122}$$
$$\dot{W} = -Y L_I(p) \tag{123}$$

with Y and Φ from (99) and (101), and with

$$W = \begin{bmatrix} w_1 & \dots & w_n \end{bmatrix}. \tag{124}$$

Defining E and Z as in (102)–(103) we obtain

$$\dot{E}\mathbf{1} = Z\mathbf{1} - \dot{Y}\mathbf{1} = Z\mathbf{1} - \gamma\big[\Phi(p)\mathbf{1} - Y\mathbf{1}\big] = -\gamma E\mathbf{1} + Z\mathbf{1} \tag{125}$$
$$\dot{W}\mathbf{1} = 0\,. \tag{126}$$

From (23) we have $L_P \equiv L_P S S^T$ and $L_I \equiv L_I S S^T$, which means we can multiply both sides of (122)–(123) from the right by S to obtain

$$\dot{Y}S = -YS\big[\gamma I + L_P^\star(p)\big] + WSL_I^\star(p) + \gamma\Phi(p)S \tag{127}$$
$$\dot{W}S = -YSL_I^\star(p)\,. \tag{128}$$

With the change of variables

$$H = WS + \gamma\Phi(p)S\big[L_I^\star(p)\big]^{-1} \tag{129}$$
$$\Omega = \begin{bmatrix} YS & H \end{bmatrix} \tag{130}$$

the equations (127)–(128) become

$$\dot{\Omega} = \Omega F^T + N G^T\,, \tag{131}$$

where

$$F = \begin{bmatrix} -\gamma I - L_P^\star(p) & L_I^\star(p) \\ -L_I^\star(p) & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{132}$$
$$N = \gamma Z S\big[L_I^\star(p)\big]^{-1} + \gamma\Phi(p)S\frac{d}{dt}\big[L_I^\star(p)\big]^{-1}. \tag{133}$$

We will write N as the sum

$$N = \gamma \sum_{i=0}^{n} N_i \tag{134}$$

where

$$N_0 = ZS\big[L_I^\star(p)\big]^{-1} \tag{135}$$
$$N_i = -\Phi(p)S\big[L_I^\star(p)\big]^{-1} \sum_{k=1}^{m} \frac{\partial L_I^\star(p)}{\partial p_i(k)} \big[L_I^\star(p)\big]^{-1} \dot{p}_i(k) \quad \text{for } 1 \leqslant i \leqslant n \tag{136}$$

and $p_i = [p_i(1) \, \dots \, p_i(m)]^T \in \mathbb{R}^m$. We now derive bounds on these matrices N_i. First, using (37) we obtain

$$N_0 N_0^T = ZS\big[L_I^\star(p)\big]^{-2} S^T Z^T \leqslant \frac{1}{\varepsilon^2} Z S S^T Z^T \tag{137}$$

Next, using (30) and our assumption that $p(t) \in \mathfrak{C}$ for all $t \geqslant t_0$, we obtain

$$\begin{aligned} |\Phi(p)S|_F^2 &= \Big|\big[\Phi(p) - \phi(p_i)\mathbf{1}^T\big]S\Big|_F^2 \leqslant \big|\Phi(p) - \phi(p_i)\mathbf{1}^T\big|_F^2 \\ &\leqslant \sum_{j \neq i} |\phi(p_j) - \phi(p_i)|^2 \\ &\leqslant (n-1)\mathfrak{a}(\mathfrak{d}(n))\zeta(p_i)\,. \end{aligned} \tag{138}$$

It follows from (37) and the fact that b has bounded partial derivatives that there exists a constant $k > 0$ such that

$$|N_i|_F^2 \leqslant k\zeta(p_i)|\dot{p}_i|^2 \tag{139}$$

for $1 \leqslant i \leqslant n$. This constant k depends on n, ε, and the bounds on the partial derivatives of b. Let σ be a constant such that $0 < \sigma < 1$ and

$$\sigma \leqslant \frac{\varepsilon(\gamma + \rho)}{(\gamma + \bar{\rho}^2) + 2\varepsilon\bar{\varepsilon}}\,. \tag{140}$$

Then the positive definite matrices

$$P = \begin{bmatrix} I & -\sigma I \\ -\sigma I & I \end{bmatrix}, \quad Q = \begin{bmatrix} (\gamma + \rho)I & 0 \\ 0 & \sigma\varepsilon I \end{bmatrix} \tag{141}$$

satisfy

$$(1 - \sigma)I \leqslant P \leqslant (1 + \sigma)I \tag{142}$$

and

$$\begin{aligned} &PF + F^T P + Q = \\ &\begin{bmatrix} -2L_P^\star(p) + (\rho - \gamma)I + 2\sigma L_I^\star(p) & \sigma\gamma I + \sigma L_P^\star(p) \\ \sigma\gamma I + \sigma L_P^\star(p) & -2\sigma L_I^\star(p) + \sigma\varepsilon I \end{bmatrix} \\ &\leqslant -\sigma \cdot \underbrace{\begin{bmatrix} \big(\frac{1}{\sigma}(\gamma + \rho) - 2\bar{\varepsilon}\big)I & -\gamma I - L_P^\star(p) \\ -\gamma I - L_P^\star(p) & \varepsilon I \end{bmatrix}}_{R(p)} \leqslant 0 \end{aligned} \tag{143}$$

because (140) implies that $R(p) \geqslant 0$. Let $\kappa > 0$ be such that

$$\kappa < \min\Big\{ \frac{\gamma+\rho}{\sigma},\, \sigma\varepsilon \Big\}\,. \tag{144}$$

Then we have

$$\begin{aligned} PGG^{T}P &= \begin{bmatrix} \sigma^2 I & -\sigma I \\ -\sigma I & I \end{bmatrix} \\ &= P - \begin{bmatrix} (1-\sigma^2)I & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{145}$$

and thus also

$$\begin{aligned} &Q - \frac{\kappa}{1+\sigma} PGG^{T}P \\ &\quad = \begin{bmatrix} \big[\gamma+\rho+\kappa(1-\sigma)\big]I & 0 \\ 0 & \sigma\varepsilon I \end{bmatrix} - \frac{\kappa}{1+\sigma} P \\ &\quad \geqslant \min\big\{\gamma+\rho+\kappa(1-\sigma),\, \sigma\varepsilon\big\} I - \kappa I = \alpha I\,, \end{aligned} \tag{146}$$

where $\alpha = \min\{\gamma+\rho-\sigma\kappa,\, \sigma\varepsilon-\kappa\}$. It now follows that

$$PF + F^{T}P + \frac{\kappa}{1+\sigma} PGG^{T}P \leqslant -\alpha I\,. \tag{147}$$

We define the matrix

$$\Psi = \Omega P \Omega^{T} + \beta E \mathbf{1}\mathbf{1}^{T} E^{T} + W \mathbf{1}\mathbf{1}^{T} W^{T} \tag{148}$$

where $\beta > 0$ is a constant parameter. Defining

$$\xi = \frac{\gamma^2 (n+1)(\sigma+1)}{\kappa}\,, \tag{149}$$

we use (23), (125), (126), (131), (134), (137), and (147) to obtain

$$\begin{aligned} \dot{\Psi} &= \Omega\big[PF + F^{T}P\big]\Omega^{T} + \gamma \sum_{i=0}^{n} \big[N_i G^{T} P \Omega^{T} + \Omega P G N_i^{T}\big] \\ &\qquad - 2\beta\gamma E\mathbf{1}\mathbf{1}^{T}E^{T} + \beta Z\mathbf{1}\mathbf{1}^{T}E^{T} + \beta E\mathbf{1}\mathbf{1}^{T}Z^{T} \qquad (150) \\ &\leqslant \Omega\big[PF + F^{T}P + \frac{\kappa}{1+\sigma} PGG^{T}P\big]\Omega^{T} - \beta\gamma E\mathbf{1}\mathbf{1}^{T}E^{T} \\ &\qquad + \xi \sum_{i=0}^{n} N_i N_i^{T} + \frac{\beta}{\gamma} Z\mathbf{1}\mathbf{1}^{T}Z^{T} \qquad (151) \\ &\leqslant -\alpha \Omega\Omega^{T} - \beta\gamma E\mathbf{1}\mathbf{1}^{T}E^{T} \\ &\qquad + \max\Big\{\frac{n\beta}{\gamma}, \frac{\xi}{\varepsilon^2}\Big\} ZZ^{T} + \xi \sum_{i=1}^{n} N_i N_i^{T}\,. \qquad (152) \end{aligned}$$

Because $ES = -YS$ we also have

$$\Omega\Omega^{T} = YSS^{T}Y^{T} + HH^{T} = ESS^{T}E^{T} + HH^{T} \tag{153}$$

and therefore

$$\dot{\Psi} \leqslant -\nu_1 EE^{T} - \alpha HH^{T} + \nu_2\, ZZ^{T} + \xi \sum_{i=1}^{n} N_i N_i^{T}\,, \tag{154}$$

where

$$\nu_1 = \min\big\{\alpha,\, n\beta\gamma\big\} \tag{155}$$

$$\nu_2 = \max\Big\{\frac{n\beta}{\gamma}, \frac{\xi}{\varepsilon^2}\Big\}\,. \tag{156}$$

Defining the storage function $U = \mathrm{Tr}(\Psi)$ we see that

$$\begin{aligned} \dot{U} \leqslant -\alpha |H|_F^2 + \sum_{i=1}^{n} \Big[&-\nu_1 |e_i|^2 \\ &+ \nu_2 |z_i|^2 + \xi k \zeta(p_i) |\dot{p}_i|^2 \Big]\,. \end{aligned} \tag{157}$$

Furthermore, we can use (113) to bound (121) from above as

$$\begin{aligned} \dot{V} \leqslant \sum_{i=1}^{n} \Big[&-\dot{p}_i^{T}\big[B + B^{T}\big]\dot{p}_i - 2c\zeta(p_i)|\dot{p}_i|^2 \\ &- \frac{1}{2} z_i^{T}\big[\Lambda + \Lambda^{T}\big] z_i + 2e_i^{T}\Gamma\big[\Lambda + \Lambda^{T}\big]^{-1}\Gamma e_i \Big]\,. \end{aligned} \tag{158}$$

We assume that (38) holds with

$$\delta_1 = \frac{1}{2}\sqrt{\frac{\nu_1}{\nu_2}}\,, \qquad \delta_2 = \frac{4\delta_1 \nu_2}{\xi k}\,, \tag{159}$$

and we choose $\mu > 0$ so that

$$\Gamma < \frac{\mu\nu_1}{2\delta_1} I < \delta_1\big[\Lambda + \Lambda^{T}\big]\,. \tag{160}$$

Upon taking inverses and then multiplying by Γ from the left and the right, we obtain

$$\frac{1}{\delta_1}\Gamma\big[\Lambda + \Lambda^{T}\big]^{-1}\Gamma < \Gamma\,. \tag{161}$$

In particular, (160) and (161) imply the existence of a scalar constant $\nu > 0$ such that

$$\frac{1}{2}\big[\Lambda + \Lambda^{T}\big] - \mu\nu_2 I \geqslant \nu I \tag{162}$$

$$\mu\nu_1 I - 2\Gamma\big[\Lambda + \Lambda^{T}\big]^{-1}\Gamma \geqslant \nu I\,. \tag{163}$$

Furthermore, (38) and (160) imply

$$2c \geqslant \mu\xi k\,. \tag{164}$$

We then define $\Upsilon(p, \dot{p}, E, W) = V + \mu U$ and use (157), (158), (162), (163), and (164) to obtain

$$\dot{\Upsilon} \leqslant -\alpha\mu |H|_F^2 + \sum_{i=1}^{n} \Big[-\dot{p}_i^{T}\big[B + B^{T}\big]\dot{p}_i - \nu|z_i|^2 - \nu|e_i|^2 \Big]\,.$$

The rest of the proof mimics the proof of Theorem 3 above.

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